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ALMOST CONGRUENT TRIANGLES

ROBERT T. JONES and BRUCE B. PETERSON, Middlebury College

1. Introduction. Perhaps nothing is more familiar from high school geometry than the collection of congruence theorems for triangles we denote by SAS (two sides and the included angle), ASA, SSS, and AAS. The curious student, of course, is expected to wonder aloud about the other two combinations of three letters from among three S's and three A's. Hopefully, he leads himself to the discovery that AAA does not determine the triangle in the sense of congruence and thereby invents similarity. The final combination, SSA, causes even more difficulty since it is sometimes a condition for congruence and sometimes not. It is seldom noticed that it is never a condition for similarity without congruence.

Unfortunately, these theorems are often viewed as the last word on congruence and lead many to believe they understand how the various parts of a triangle determine it. In fact remarkably few seem aware that even knowledge of five parts of a triangle does not necessarily determine it in the sense of congruence. The problem of determining a triangle given five of its parts has apparently been around for some time, but despite its familiarity, we have been able to locate only four brief references. For these we are indebted to Murray Klamkin. We would appreciate hearing from others who may know of further sources. Our purposes here are to explicate the situation and to discuss some natural questions which follow from it. The misunderstanding may flow from the fact that the formulae (SAS, ASA, etc.) carry information about adjacency which is critical for congruence. We ask here about determination of a triangle given only information about magnitudes of angles and sides.

For our purposes, a "part" of a triangle is a side or an angle. In the cases of both congruence and similarity we will ignore differences in orientation.

2. An analytic proof. Since they must include either three angles or three sides, five parts always determine a triangle up to similarity or congruence. We will show that there exist similar incongruent triangles sharing two sides. Professional mathematicians we have approached often prove this fact by a simple application of the intermediate value theorem for continuous functions. We include it here as an example of mathematical overkill and because this esthetic flaw led us to deeper consideration of the geometric problems involved.

Proof. Let A be any point other than the center in the interior of a circle of radius r. Let $B'C$ be the chord through A perpendicular to the diameter through A. Let $C'B$ be any other chord through A with C' on the same side of the diameter through A as B' and lying "above" $B'C$. Let $\alpha = \measuredangle B'AC' = \measuredangle BAC$.

Since $\measuredangle AB'C'$ and $\measuredangle ABC$ are inscribed angles measured by the same arc, they are equal. Call this angle β and the remaining angles γ. The triangles ABC and $AB'C'$ are clearly similar and have a pair of equal sides ($B'A = AC$). Now let $x(\alpha) = AC'$ and $y(\alpha) = BC$. Both these functions are continuous, and $x(\alpha) > 0$ for $0 \leqq \alpha \leqq \pi/2$.

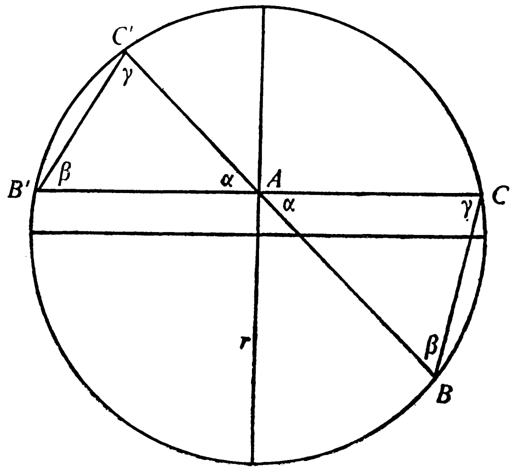

FIG. 1.

Hence for $0 \leqq \alpha \leqq \pi/2$, the function $f(\alpha) = y(\alpha)/x(\alpha)$ is continuous. Moreover $f(0) = 0$ and $f(\pi/2) > 1$. By the intermediate value theorem there is an angle α_0, $0 < \alpha_0 < \pi/2$, for which $f(\alpha_0) = 1$ and $y(\alpha_0) = x(\alpha_0)$.

It only remains to show that the triangles ABC and $AB'C'$, are not congruent. If they were, then by looking at angles opposite pairs of equal sides we find that $\beta = \gamma$ and $\alpha = \beta$, and both triangles are equilateral. But by construction we must have $x < AB'$. This contradiction establishes the result.

3. A geometric argument. The analytic proof is disappointing for several reasons. It does not provide a simple way of constructing two triangles of the desired types. Moreover, since they have opposite orientation, the triangles it produces are only similar in the plane if one allows reflections. Of course this is not a real problem since we can easily reflect one of the triangles and get a more satisfying similar pair. Although we will ignore the questions of embedding and similarity in the plane versus space, the orientation is at least an esthetic annoyance.

More important is the matter of excess machinery. Why should such a simply stated problem require considerations of continuity? In fact it does not.

To see this let a and b be real numbers and γ the angle between sides with lengths a and b. This prescription determines a triangle with a third side c. We will assume without loss of generality that $a \leqq b \leqq c$. Now construct a new triangle with γ between sides of length b and c and third side d. If these triangles are similar, we must have

(1) $$\frac{a}{b} = \frac{b}{c}.$$

Naturally we must also have

(2) $$a + b > c.$$

Noting that condition (1) is also sufficient for similarity we can easily construct the desired triangles. First choose real numbers a, b, and c with $a \leqq b \leqq c$ subject to conditions (1) and (2) and construct a triangle with sides a, b, and c. Let γ be the angle opposite side c. Now construct a new triangle with sides b and c, including angle γ. Or, equivalently, construct a triangle with sides b, c, and $d = c^2/b$.

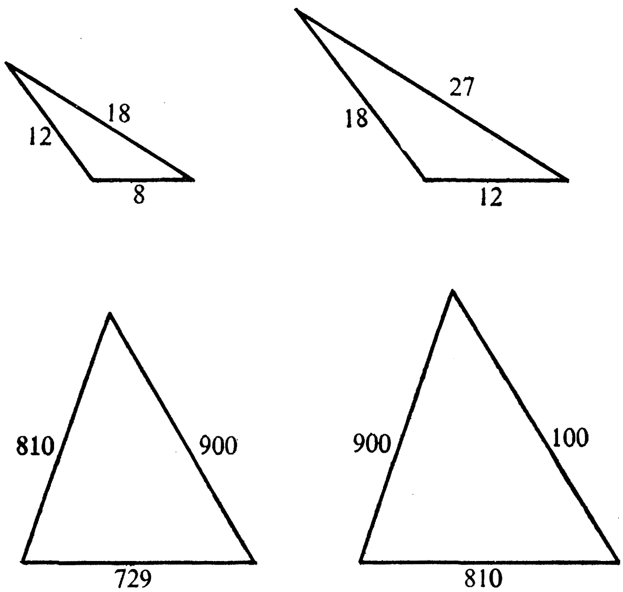

FIG. 2.

Of course if $a = b$ or $b = c$, the triangles are in fact equilateral and congruent. Otherwise $a < b < c < d$, and the triangles are incongruent.

4. The shape of an almost congruent triangle. Two incongruent similar triangles which share five parts will be called **almost congruent**. Since whether a given triangle can belong to an almost congruent pair can be determined without considering the other member of the pair, it makes sense to speak of *a single almost congruent triangle*. Almost congruence of a triangle is preserved by similarity. Clearly conditions (1) and (2) are sufficient to guarantee almost congruence of a triangle. S.nce not every triangle satisfies these conditions; it seems reasonable to ask what may be said about the shape of an almost congruent triangle.

For simplicity, all triangles will be denoted by triples (a, b, c) where a, b, and c are the side lengths, and it is assumed that $a \leqq b \leqq c$. The angles opposite the sides will be denoted by the corresponding Greek letters, so that $\alpha \leqq \beta \leqq \gamma$. For any almost congruent triangle the number $R = a/b = b/c$ is called the common ratio.

THEOREM 1. *For any almost congruent triangle,* $\dfrac{-1+\sqrt{5}}{2} < R < 1.$

Proof. The second inequality is trivial since $a < b$. The triangle (a, b, c) is similar to the triangle $(R, 1, 1/R)$, so we must have $R + 1 > 1/R$ and $R^2 + R - 1 > 0$. It follows that $R > (-1 + \sqrt{5})/2$.

In fact, for any R satisfying these inequalities, there is a unique (up to similarity) almost congruent triangle $(R, 1, 1/R)$.

If $(R, 1, 1/R)$ is an almost congruent triangle, then $1/R^2 = 1 + R^2 - (2R)\cos\gamma$ and $\cos\gamma = (R^4 + R^2 - 1)/2R^3$. It is easily verified that the function $g(R) = (R^4 + R^2 - 1)/2R^3$, defined for $(-1 + \sqrt{5})/2 \leqq R \leqq 1$, has range $[-1, \frac{1}{2}]$ and is increasing. Hence the angle γ (the maximum angle in our triangle) is a decreasing function of R, and for each γ with $(\pi/3) < \gamma < \pi$, there is a unique R such that the maximum angle in the triangle $(R, 1, 1/R)$ is γ.

Similarly, the minimum angle α is an increasing function of R, and for each α with $0 < \alpha < (\pi/3)$, there is a unique R such that the minimum angle in the triangle $(R, 1, 1/R)$ is α. The third angle β is an increasing function of R with range $(0, \pi/3)$, and each β with $0 < \beta < (\pi/3)$ determines a unique R such that the triangle $(R, 1, 1/R)$ has intermediate angle β.

An almost congruent right triangle must have $R^2 + 1 = 1/R^2$ or $R^4 + R^2 - 1 = 0$. Hence $R = \sqrt{(-1 + \sqrt{5})/2}$. Denoting this value by R_0, we see that acute almost congruent triangles are characterized by common ratios $R > R_0$, and obtuse almost congruent triangles are characterized by common ratios $R < R_0$.

5. Where to find almost congruent triangles. There are two interesting ways in which almost congruent triangles can be found. The first involves a family of spirals.

Pick R with $(-1 + \sqrt{5})/2 < R < 1$ and $0 < \alpha < (\pi/3)$ with $R^2 = 1 + 1/R^2 - (2/R)\cos\alpha$. Then $(R, 1, 1/R)$ is an almost congruent triangle with minimum angle α. Now consider the spiral with polar equation $r = R^{-\theta/\alpha}$. Let φ be any angle and consider the points on the curve determined by the directions φ and $\varphi + \alpha$: $(R^{-\varphi/\alpha}, \varphi)$, $(R^{-(\varphi/\alpha)-1}, \varphi + \alpha)$. The distance between these points is given by

$$d = \sqrt{[R^{-\varphi/\alpha}\cos\varphi - R^{-(\varphi/\alpha)-1}\cos(\varphi + \alpha)]^2 + [R^{-\varphi/\alpha}\sin\varphi - R^{-(\varphi/\alpha)-1}\sin(\varphi + \alpha)]^2}$$

$$= R^{-\varphi/\alpha}\sqrt{1 + R^{-2} - 2R^{-1}\cos(\varphi + \alpha - \varphi)} = R^{-\varphi/\alpha}\sqrt{R^2} = R^{-\varphi/\alpha+1}.$$ Obviously $R = R^{-\varphi/\alpha+1}/R^{-\varphi/\alpha} = R^{-\varphi/\alpha}/R^{-\varphi/\alpha-1}$.

Hence, any two radius vectors including an angle α intersect the spiral in points which, together with the origin, determine an almost congruent triangle with common ratio R. Any two such triangles sharing a radius vector form an almost congruent pair. (See Figure 3.)

Since α is an increasing function of R, the function $g(R) = (-\ln R)/\alpha$ is also decreasing for $(-1 + \sqrt{5}/2) < R < 1$ with range $(0, +\infty)$. Hence for each real $k > 0$, there is a unique R (and a unique α) such that $k = (-\ln R)/\alpha$. In other words, every exponential spiral $r = e^{k\theta}$ determines almost congruent triangles for some angle α.

If α is allowed to take on its extreme values 0 and $\pi/3$, the corresponding equations

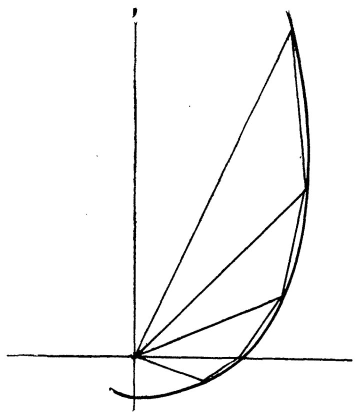

FIG. 3.

for R become $R^2 = 1 + (1/R^2) - (2/R)$ and $R^2 = 1 + (1/R^2) - (1/R)$. The first of these reduces to $R^4 - R^2 + 2R - 1 = (R^2 + R - 1)(R^2 - R + 1) = 0$ whose only nonnegative solution is $R = (-1 + \sqrt{5})/2$. This value of R determines the degenerate almost congruent triangle $(-1 + \sqrt{5})/2$, 1, $(1 + \sqrt{5})/2$, a line segment. The corresponding spiral degenerates to a line. Not surprisingly the second equation has $R = 1$ as its sole nonnegative root, and determines the triangle $(1, 1, 1)$ which is degenerate as an almost congruent triangle since any other triangle sharing five of its parts must be congruent to it. The corresponding spiral degenerates to a circle of radius 1.

It is easy to construct arbitrarily long sequences of almost congruent triangles sharing the common ratio R (and consequently the minimum angle α), so that each consecutive pair in the sequence is an almost congruent pair. First construct one almost congruent triangle (a, b, c) with minimum angle α and extend sides b and c. (See Figure 4.) Let the vertices of this triangle be A, B and C with A opposite a, etc. At B construct a line making angle γ with c, and let this line intersect the extension of b at D. Let $AD = d$ and $BD = x$.

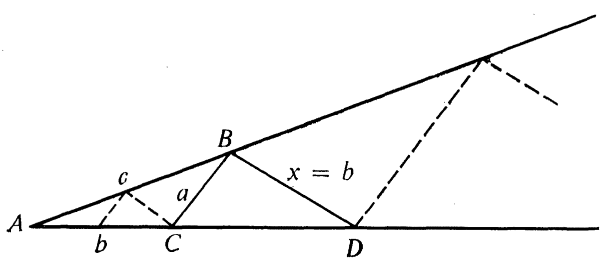

FIG. 4.

From the almost congruence of (a, b, c), it follows that (x, c, d) is also almost congruent. We claim that the triangles are almost congruent to each other.

To see this note that $a/b = x/c$ from the similarity and that $a/b = b/c$ and $x/c = c/d$ from the almost congruence of the two triangles. Hence $b/c = c/d$ and

$d = c^2/b$. Therefore

$$x = \sqrt{c^2 + d^2 - 2cd\cos\alpha} = \sqrt{c^2 + \frac{c^4}{b^2} - 2\left(\frac{c^3}{b}\right)\cos\alpha}$$

$$= \frac{c}{b}\sqrt{b^2 + c^2 - 2bc\cos\alpha} = \frac{c}{b}\cdot a = b,$$

and the triangles are almost congruent to each other.

Of course we can continue this process indefinitely, and we could as well start at C and work within the original triangle.

6. Almost congruent triangles with integral sides. Since we are all Pythagoreans at heart, we could not leave this subject without considering whether almost congruent triangles or almost congruent pairs of triangles exist with exclusively integral sides. Clearly there are no almost congruent right triangles with integral sides because the common ratio R_0 is irrational. In fact there are individual triangles and almost congruent pairs with integral sides for every rational ratio $(-1 + \sqrt{5})/2 < R < 1$. Moreover, we can characterize them completely with the following theorems:

THEOREM 2. *Let (a, b, c) be a triple of integers, and let $R = a/b$. Then (a, b, c) is an almost congruent triangle if and only if*

(I) $\dfrac{-1 + \sqrt{5}}{2} < R < 1$ *and*

(II) *there exist positive integers k, p, q, with $p < q$, p and q relatively prime, and $a = kp^2$, $b = kpq$, $c = kq^2$.*

Proof. Suppose first that (I) and (II) hold. Then certainly $a < b < c$ and $a/b = b/c$. The first inequality in (I) implies that

$$R + 1 > \frac{-1 + \sqrt{5}}{2} + 1 = \frac{1 + \sqrt{5}}{2} = 1 \bigg/ \left(\frac{-1 + \sqrt{5}}{2}\right) > \frac{1}{R}.$$

Since $R = a/b = b/c$ we have $a/b + 1 > c/b$ and $a + b > c$. Hence (a, b, c) is an almost congruent triangle.

Now suppose we know that (a, b, c) is an almost congruent triangle. Condition (I) has already been proved in Theorem 1.

The ratio $R = a/b = b/c$ is rational, so we can find relatively prime integers p and q with $p < q$ such that $R = p/q$. From $aq = bp$ we see that p divides a and there is an integer k_1 such that $a = k_1 p$. Then $b = aq/p = k_1 q$. Similarly, from $bq = cp$ we can deduce the existence of an integer k_2 such that $b = k_2 p$ and $c = k_2 q$. Since p divides $b = k_1 q$, in fact p divides k_1 and there is an integer m_1 with $k_1 = m_1 p$. Similarly there is an integer m_2 with $k_2 = m_2 q$. We have the following equations:

(1) $$a = k_1 p$$

(2) $$b = k_1 q = k_2 p$$

(3) $$c = k_2 q$$

(4) $$k_1 = m_1 p$$

(5) $$k_2 = m_2 q.$$

Substituting (4) and (5) in (2) we have $m_1 pq = m_2 qp$ and $m_1 = m_2$. Substituting (4) and (5) in (1), (2), and (3), we have

$$a = k_1 p = m_1 p^2$$

$$b = k_1 q = m_1 pq$$

$$c = k_2 q = m_1 q^2.$$

Taking $k = m_1$ completes the proof.

THEOREM 3. *Let (a, b, c, d) be a quadruple of integers and $R = a/b$. Then (a, b, c) and (b, c, d) are an almost congruent pair if and only if*

(I) $\dfrac{-1+\sqrt{5}}{2} < R < 1$ *and*

(II) *there exist positive integers k, p, q, with $p < q$, p and q relatively prime, and $a = kp^3$, $b = kp^2q$, $c = kpq^2$, and $d = kq^3$.*

Proof. If the conditions hold, then (a, b, c) is almost congruent by Theorem 2. Condition (II) implies that $R = a/b = b/c = c/d$ and (a, b, c) and (b, c, d) are an almost congruent pair.

Now suppose we know that (a, b, c) and (b, c, d) are an almost congruent pair. Then since (a, b, c) is almost congruent, condition (I) is satisfied, and there are integers n, p, q with $p < q$, p and q relatively prime, and $a = np^2$, $b = npq$ and $c = nq^2$. Hence $a/b = p/q$, and since $a/b = c/d$ we have $qc = pd$. This implies that p divides $c = nq^2$ and p divides n. Hence there is an integer k such that $n = kp$ and $a = kp^3$, $b = kp^2q$, $c = kpq^2$ and $d = (qc/p) = (qkpq^2/p) = kq^3$. This completes the proof.

COROLLARY. *There exist arbitrarily long sequences $\{T_1, T_2, \cdots, T_n\}$ of almost congruent triangles with integral sides such that $\{T_i, T_{i+1}\}$ is an almost congruent pair for each i.*

Proof. Choosing $R = (p/q) > (-1 + \sqrt{5})/2$, the sequence $\{p^{n+1}, p^n q, p^{n-1}q^2, \cdots, pq^n, q^{n+1}\}$ may be used to construct the triangles.

A triangle with relatively prime integral sides will be called **primitive.** It is now a simple matter to construct primitive almost congruent triangles and pairs.

First, suppose that $p = 1$. Condition (I) implies that $(1/q) > (-1 + \sqrt{5})/2$ and $q < (1 + \sqrt{5})/2$. But then $q = 1$, and $R = 1$. Hence there is no primitive almost congruent triangle with $p = 1$.

If $p = 2$, then $(2/q) > (-1 + \sqrt{5})/2$ and $q < 1 + \sqrt{5} < 4$. Hence $q = 3$. The primitive almost congruent triangle with common ratio $R = 2/3$ is (4, 6, 9). The corresponding primitive almost congruent pair is given by the quadruple (8, 12, 18, 27).

Table 5 lists all the primitive almost congruent triangles and pairs for values of $p \leqq 12$. They have been divided into acute and obtuse triangles by means of the relationships in section 4.

Since $\frac{3}{4} < R_0$, the triangle with $R = \frac{3}{4}$ is obtuse. But since $\frac{4}{5} > R_0$ and $(p/p+1) > (p-1/p)$ for all positive integers p, there is an acute primitive almost congruent triangle with $a = p^2$ for each integer $p \geqq 4$, namely the triangle $(p^2, p(p+1), (p+1)^2)$.

It is slightly more complicated to see that for each positive integer $p \geqq 2$, there is an almost congruent obtuse triangle with integral sides and $a = p^2$. To find one it will be sufficient to produce a positive integer q with $R_0^2 = (-1+\sqrt{5})/2 < (p/q) < R_0$ or $(p/R_0^2) > q > (p/R_0)$. There is such a q if $(p/R_0^2) - (p/R_0) > 1$. But

$$\frac{1}{R_0{}^2} - \frac{1}{R_0} = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}} > 1.6 - 1.3 = .3 > \frac{1}{p}$$

for all $p \geqq 4$. The almost congruent triangle (p^2, pq, q^2) is obtuse, but, of course, we cannot necessarily guarantee that it is primitive. The cases for $p = 2$ and 3 have been settled by example.

TABLE 5.

PRIMITIVE ALMOST CONGRUENT TRIANGLES AND PAIRS

p	maximum q for p	R	Acute triangles and pairs	Obtuse triangles and pairs
2	3	2/3		(4, 6, 9) (8, 12, 18, 27)
3	4	3/4		(9, 12, 16) (27, 36, 48, 64)
4	6	4/5	(16, 20, 25) (64, 80, 100, 125)	
5	8	5/6	(25, 30, 36) (125, 150, 180, 216)	
		5/7		(25, 35, 49) (125, 175, 245, 343)
		5/8		(25, 40, 64) (125, 200, 320, 512)
6	9	6/7	(36, 42, 49) (216, 252, 294, 343)	
7	11	7/8	(49, 56, 64) (343, 392, 448, 512)	
		7/9		(49, 63, 81) (343, 441, 567, 729)
		7/10		(49, 70, 100) (343, 490, 700, 1000)
		7/11		(49, 77, 121) (343, 539, 847, 1331)

TABLE 5. (Cont'd)

PRIMITIVE ALMOST CONGRUENT TRIANGLES AND PAIRS

p	maximum q for p	R	Acute triangles and pairs	Obtuse triangles and pairs
8	12	8/9	(64, 72, 81) (512, 576, 648, 729)	
		8/11		(64, 88, 121) (512, 704, 968, 1331)
9	14	9/10	(81, 90, 100) (729, 810, 900, 1000)	
		9/11	(81, 99, 121) (729, 891, 1089, 1331)	
		9/13		(81, 117, 169) (729, 1053, 1521, 2197)
		9/14		(81, 126, 196) (729, 1134, 1764, 2744)
10	16	10/11	(100, 110, 121) (1000, 1100, 1210, 1331)	
		10/13		(100, 130, 169) (1000, 1300, 1690, 2197)
11	17	11/12	(121, 132, 144) (1331, 1452, 1584, 1728)	
		11/13	(121, 143, 169) (1331, 1573, 1859, 2197)	
		11/14		(121, 154, 196) (1331, 1694, 2156, 2744)
		11/15		(121, 165, 225) (1331, 1815, 2475, 3375)
		11/16		(121, 176, 256) (1331, 1936, 2816, 4096)
		11/17		(121, 187, 289) (1331, 2057, 3179, 4913)
12	19	12/13	(144, 156, 169) (1728, 1872, 2028, 2197)	
		12/17		(144, 204, 289) (1728, 2448, 3468, 4913)
		12/19		(144, 228, 361) (1728, 2736, 4332, 6859)

7. Determining a triangle by its parts. Let us ask again in what sense a triangle is determined by some combination of its parts. In other words, given a collection of angles and side lengths, what is the maximum number of incongruent triangles which may share those parts. We have seen that two incongruent triangles may share five parts. It is easy to see that two is the maximum.

If four parts are given we have in fact only one new case to consider. If the four parts include three sides, there is a single triangle. If the four parts are two sides and two angles, we automatically have all three angles and can construct at most two triangles by the previous paragraph. The remaining situation involves a single side and all three angles. In this case the triangle is determined up to similarity. There are exactly three (depending upon the placement of the side) triangles. Hence the maximum, given four parts, is three.

If only two parts are given, the number of possible triangles is unlimited. We cannot even be assured of similarity except in the case where both parts are angles.

If only three parts are given, we can of course have infinitely many triangles when all three parts are angles. If two parts are angles and one a side, we have a previous case and at most three triangles. If all three parts are sides, there is but a single triangle. If two sides and an angle are given, we can find up to four incongruent triangles, depending upon the placement of the angle. If it lies between the sides, there is a single triangle. If it is adjacent to the shorter side, there is a single triangle. If it is adjacent to the longer side, there may be 0, 1, or 2 triangles. In Figure 6, we have illustrated four incongruent triangles sharing two sides and an angle.

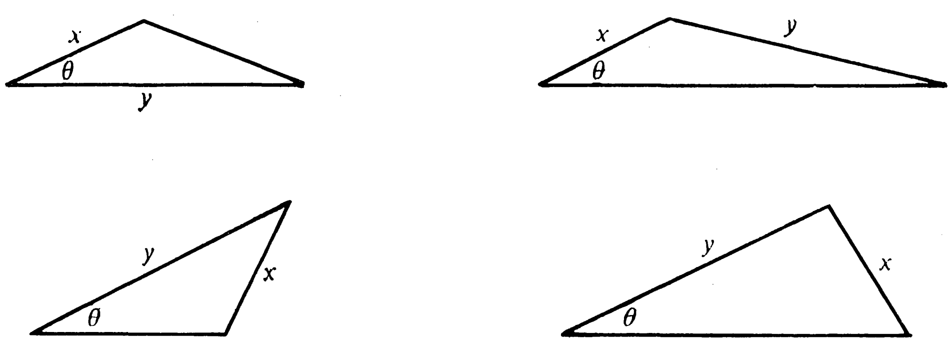

FIG. 6.

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A TWO-MOVE GAME

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Problem statement. Aristotle O'Toole proposed the following problem [**1**]. A and B play a two-stage game. In the first stage, each player covertly enters four nonnegative numbers totaling unity into the cells of a 2×2 matrix. Say A plays $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, and B, $\begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$. The entries are then exposed, and a third 2×2 matrix M is constructed, whose entries are $x_i - y_i$. This latter matrix is employed in the second stage of play as the payoff matrix of an ordinary 2×2 matrix game. The entries of M are considered payoffs to A, who chooses a column and tries to maximize, while B chooses a row and minimizes. The expected payoff to A will be denoted Val(M). We shall call the entire two-stage process G^2.

O'Toole notes that because of the symmetry of the play, it is intuitive that G^2 is a fair game, even though any particular play of the second stage game defined by M may not be. He challenges the reader to determine optimal strategies.

In this paper, we solve this problem fully; discuss some of its implications; and propose a generalization which seems to be natural but much more difficult.

Classification. In one ingenious simple stroke, O'Toole has posed a problem that touches a considerable number of advanced aspects of game theory. G^2 obviously has an infinite set of pure strategies. The concept of "value" for such a game requires a careful definition of the allowed sets of strategy mixes. Examples of possible definitions are given by Luce and Raiffa [**2**], p. 449. Although we shall be able to describe the nature of the optimal strategy mixes quite precisely verbally, it is not easy to show how they fit into any of the standard schemata. To be specific we *cannot* characterize the solution as a probability distribution over the set of admissible pure strategies corresponding to a region of E_3. Thus, in G^2, the very concepts of value and optimal strategies are quite complicated.

Note next that G^2 can be considered a game of partitioning. Each player has certain limited assets, totaling 1, which he must allocate to the four cells of M. In this sense, G^2 is something akin to the Blotto games [**3, 4**]. However a review of the literature of such games has failed to reveal any directly applicable theorems. The difficulty is that the payoff in G^2 does not depend on the individual results of the allocations in the separate cells of M, i.e., the values of $x_i - y_i$; rather it depends on the interactions among the various entries of M in that complex way characteristic of all matrix games. By choosing the simplest possible case, the 2×2 game, O'Toole has brought the problem (just barely) within the reach of this writer's solution capabilities. In the generalization I propose later, I am not so kind to the current reader.

In that there are two stages, and the second is a matrix game of the usual type, G^2 resembles a recursive game [**6**]. This resemblance is only partial. In a recursive game, the number of possible games that may be played at the second (or later) stage is finite. The first move defines a probability distribution over that finite set.

The game is randomly selected, based on this distribution. In G^2 there is no distribution over the set of possible games; or more precisely, the particular M matrix to be used in stage two is determined with certainty, rather than probabilistically. It is still possible to cast this as in a recursive game, since one can define a degenerate distribution with weight 1 attached to one point in the space of adinitted M adinitted and zero elsewhere. However the space of M-matrices is infinite, not finite, and this difference disqualifies G^2 as a recursive game, and renders recursive game theory inapplicable.

Games with infinite strategy sets have been studied widely, the most familiar case, perhaps, being games on the unit square [7, 8]. While each player in G^2 has a continuum of strategies, G^2 itself does not reduce conveniently to such a game on the unit square. Since $x_1 + x_2 + x_3 + x_4 = 1$, we see that only three of the first stage choices are independent. We can view the first move of player A as selecting the numbers x_1, x_2 and x_3, to satisfy

$$x_i \geqq 0; \; x_1 + x_2 + x_3 \leqq 1. \tag{1}$$

That is to say, the triple (x_1, x_2, x_3) is a point of E_3 in the tetrahedron defined by (1). Thus each player, at the first stage chooses a point in a region of E_3. This is *not* the same thing as a game on the unit square, nor is it clear how it might be mapped into one while preserving continuity of the payoff function, Val(M).

Thus G^2 is tantalizingly close in structure to several solved or partially solved cases from the literature, but not enough like any of them to permit use of prior theory. We shall have to meet G^2 head on, on its own grounds.

First steps to the solution. Later on, we shall attempt to lead the reader through an argument that will make the solution appear reasonable. In this section however, we dispense with heuristics and simply state the result, and begin to prove it.

DEFINITION. *A balanced column* 4-*set is a set of four first stage strategies of the form*

$$\begin{bmatrix} s & 0 \\ 1-s & 0 \end{bmatrix}, \begin{bmatrix} 1-s & 0 \\ s & 0 \end{bmatrix}, \begin{bmatrix} 0 & s \\ 0 & 1-s \end{bmatrix}, \begin{bmatrix} 0 & 1-s \\ 0 & s \end{bmatrix} \tag{2}$$

which are to be played with equal weight in a player's strategy mix. Balanced row 4-*sets are defined analogously.*

THEOREM. *The optimal strategy for A is the following. He should play a mix of balanced column* 4-*sets with parameter s, where s is drawn from the interval* [1/3, 2/3] *either deterministically, or randomly with any probabilistic distribution whatever.*

B should similarly play a mix of balanced row 4-sets. We prove this only for A; the proof for B is symmetric. We first show that the balanced column 4-set with $s = \frac{1}{2}$ (which reduces to a balanced 2-set, since $s = 1 - s$) always assures A a non-negative payoff.

If A plays a column strategy (i.e., all assets assigned to a single column), define a *naive* strategy as the rule that at stage two, he always chooses the column in M in which, at stage one, he allocated all his resources.

We assume A plays the strategies

$$\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \tag{3}$$

with equal weight. Suppose B plays any strategy

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}. \tag{4}$$

The two possible M matrices will be

$$\begin{bmatrix} \frac{1}{2}-a & -c \\ \frac{1}{2}-b & -d \end{bmatrix} \text{ and } \begin{bmatrix} -a & \frac{1}{2}-c \\ -b & \frac{1}{2}-d \end{bmatrix}. \tag{5}$$

In either case, by playing a naive strategy, A can obtain at least

$$\min[\tfrac{1}{2}-a,\ \tfrac{1}{2}-b] \quad \text{or} \quad \min[\tfrac{1}{2}-c,\ \tfrac{1}{2}-d] \tag{6}$$

respectively. His expectation is

$$\begin{aligned} \tfrac{1}{2}\min[\tfrac{1}{2}-a, \tfrac{1}{2}-b] + \tfrac{1}{2}\min[\tfrac{1}{2}-c, \tfrac{1}{2}-d] &= \tfrac{1}{2}(\tfrac{1}{2} - \max[a,b] + \tfrac{1}{2} - \max[c,d]) \\ &= \tfrac{1}{2}(1 - \max[a,b] - \max[c,d]). \end{aligned} \tag{7}$$

But $\max[a,b]$ and $\max[c,d]$ are some two of the four numbers a, b, c, and d, which total to unity. Hence

$$\max[a,b] + \max[c,d] \leqq 1, \tag{8}$$

and therefore the expression in (7) is nonnegative.

This proves that the described strategy (3) is *an* optimal one for A. We call (3) the *basic* A-strategy. (In the future, we shall merely compute the sum of the four outcomes of the games resulting from first stage choices of (2) and (4), omitting the common multiplicative factor $\frac{1}{4}$, which is the probability of each possible outcome.)

The proof above also shows that B cannot assure nonpositive payoff unless

$$\max[a,b] + \max[c,d] = 1. \tag{9}$$

This means all of B's resources must be allocated to two cells of M. Each column of M can have at most one nonzero entry in B's first stage move, for any optimal B-strategy. By symmetry, any pure strategy in an optimal A-strategy mix can have at most one nonzero entry in any row of M. (The basic strategy meets that condition, of course.)

Diagonal 4-sets. From the symmetry of the cells of M, it is easy to recognize that if there is a column strategy in an A-optimal strategy mix, then that mix must include also the other three pure strategies of (2) with equal weight, i.e., the only column strategies for A must come in balanced 4-sets.

A similar argument shows that if the A-strategy mix includes one of the form

$$\begin{bmatrix} 0 & s \\ 1-s & 0 \end{bmatrix} \tag{10}$$

then it must also include with equal weights, the three cyclic permutations

$$\begin{bmatrix} 1-s & 0 \\ 0 & s \end{bmatrix}, \begin{bmatrix} 0 & 1-s \\ s & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} s & 0 \\ 0 & 1-s \end{bmatrix}. \tag{11}$$

This set of four strategies we call a *balanced diagonal* 4-*set*. We shall show that no optimal A-strategy mix includes such a diagonal 4-set.

For consider an A-strategy mix that includes the above four choices (10) and (11) for the stage one move. Without loss of generality, assume $s > \frac{1}{2}$. For those games, consider the outcome when B plays his basic strategy. There are two possible choices for B (row 1 and row 2), and four for A. However these eight possible pairings produce for M simply four sets like the following pair (12) with permutations of rows and/or columns:

$$\begin{bmatrix} s-\frac{1}{2} & -\frac{1}{2} \\ 0 & 1-s \end{bmatrix} \text{ and } \begin{bmatrix} s & 0 \\ -\frac{1}{2} & \frac{1}{2}-s \end{bmatrix}. \tag{12}$$

The second of these M matrices has a saddle point, and for it, $\text{Val}(M) = \frac{1}{2} - s$. The first has no saddle point, but the value will be strictly less than $\min[s-\frac{1}{2}, 1-s] \leqq s - \frac{1}{2}$. This follows from the easily proved lemma that in a 2×2 matrix game without saddle point, $\text{Val}(M)$ is strictly between min[row max] and max[col min], the former of which is presented above. Thus, the expectation to A will be (strictly) negative if he includes any balanced diagonal 4-set with $s > \frac{1}{2}$ in his strategy mix and B plays the basic strategy. The case $s < \frac{1}{2}$ is similar. The case $s = \frac{1}{2}$ is left to the reader.

Other excluded 4-sets. We have proved thus far that only balanced column 4-sets occur in an optimal A strategy mix. It remains to determine which ones. We first show that such column sets for A, with $s > 2/3$, cannot be part of a winning strategy mix. For consider such a set, and observe the result if B plays

$$\begin{bmatrix} 2/3 & 1/3 \\ 0 & 0 \end{bmatrix}. \tag{13}$$

The four M matrices will be

$$\begin{bmatrix} s-2/3 & -1/3 \\ 1-s & 0 \end{bmatrix}, \begin{bmatrix} 1/3-s & -1/3 \\ s & 0 \end{bmatrix}, \begin{bmatrix} -2/3 & s-1/3 \\ 0 & 1-s \end{bmatrix} \text{ and } \tag{14}$$

$$\begin{bmatrix} -2/3 & 2/3-s \\ 0 & s \end{bmatrix}.$$

Each of these has a saddle point. The respective values are $\min(s - 2/3,\ 1 - s)$, $-1/3$, $1 - s$, and $2/3 - s$. The sum of these is easily shown by simple algebra to be (strictly) negative.

Similarly, we can exclude $s < 1/3$. It remains only to show that all remaining balanced column 4-sets do provide a nonnegative payoff to A against any pure B strategy.

Naive strategy solutions.

LEMMA. *If* $a, b \geqq 0$, *and* $s \geqq \frac{1}{2}$, *we have*

$$\min(s - a, 1 - s - b) + \min(1 - s - a, s - b) \tag{15}$$

equal to one of the following three quantities:

$$\begin{cases} 1 - 2a & \text{(Case Ia)} \\ 1 - 2b & \text{(Case Ib)} \\ 2 - 2s - (a + b) & \text{(Case II)}. \end{cases} \tag{16}$$

Proof. The three expressions of (16) correspond to three of the four possible ways to assign the minimum in the two parentheses in (15). The fourth case in (15) would be

$$s - a < 1 - s - b \quad \text{and} \quad 1 - s - a > s - b, \tag{17}$$

which cannot occur.

Either Case Ia or Case Ib will be referred to as Case I.

Now consider a balanced column 4-set of A strategies with $1/3 \leqq s \leqq 2/3$. Without loss of generality, we can assume $s \geqq \frac{1}{2}$. Consider any B strategy

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}. \tag{18}$$

The four possible M matrices are

$$\begin{bmatrix} s - a & -c \\ 1 - s - b & -d \end{bmatrix}, \begin{bmatrix} 1 - s - a & -c \\ s - b & -d \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} -a & s - c \\ -b & 1 - s - d \end{bmatrix}, \begin{bmatrix} -a & 1 - s - c \\ -b & s - d \end{bmatrix}.$$

In each stage-two game, we consider the result if A plays a naive strategy. The sum of the payoffs will be:

$$\begin{aligned} &\min(s - a, 1 - s - b) + \min(1 - s - a, s - b) \\ &\quad + \min(s - c, 1 - s - d) + \min(1 - s - c, s - d). \end{aligned} \tag{20}$$

The lemma is now applicable separately to the first two terms of (20), and to the last two terms. Three situations may arise:

(1) Both pairs of terms are in Case I;
(2) Both pairs of terms are in Case II;
(3) One pair is in each case.

(1) If both are in Case I, then the sum (20) comes out to be

$$2 - 2(a' + c') \tag{21}$$

where a' is a or b, and c' is c or d. In any case, the parenthesis in (21) is a subset of the sum $a + b + c + d$ which equals 1. Hence the payoff (20) is nonnegative.

(2) Also, if both pairs are in Case II, then (20) comes out to be

$$4 - 4s - (a + b + c + d) = 3 - 4s > 0 \tag{22}$$

since $s \leqq 2/3$.

The only complication is if one of the column pairs is in Case I and the other in Case II. In that instance, A may have to depart from a naive strategy. (Of course, A can detect the occurrence of this situation before making his stage-two move.)

Completion of the proof. Continuing with the notation of the previous section we suppose

$$\begin{aligned} \tfrac{1}{2} &\leqq s \leqq 2/3 \\ 1 - s - b &> s - a \\ 1 - s - c &< s - d \\ 1 - s - d &< s - c. \end{aligned} \tag{23}$$

It is still necessary to consider two subcases.

(i) Suppose

$$a < 2/3. \tag{24}$$

Then a naive strategy for A will still suffice. The various minima in (20) can be determined from (23), and the naive strategy gives

$$\begin{aligned} &1 - 2a + 2 - 2s - c - d \\ &= 2 - 2s - a + b \geqq 2 - 2s - a > 0, \end{aligned} \tag{25}$$

since $s \leqq 2/3$ and $a < 2/3$.

(ii) Next suppose

$$a \geqq 2/3. \tag{26}$$

Now A must depart from his naive strategy. He will use a naive strategy in three of the four games of (19), viz., the first, third and fourth. But if the M matrix

$$\begin{bmatrix} 1 - s - a & -c \\ s - b & -d \end{bmatrix} \tag{27}$$

occurs, then A must play in the second, rather than the first column. (The reason for this is discussed later.) This choice will assure A of a return of $-\max(c, d)$ on this play of the game. The return for the 4-set will be

$$2 - 2s - c - d + s - a - \max(c, d) \tag{28}$$
$$= 2 - s - 1 + b - \max(c, d) \geqq 1/3 - \max(c, d).$$

But since $a \geqq 2/3$, and $a + b + c + d = 1$, we have $\max(c, d) \leqq 1/3$; i.e., the payoff (28) is nonnegative.

And this completes the proof.

Heuristics. I feel the reader may be forgiven if he finds his head spinning at all the cases that had to be examined. He should pity the poor author! I considered many more cases, because I started off on the wrong track. Although I correctly guessed early along that the solution consisted of balanced 4-sets for s values in a symmetric interval about $\frac{1}{2}$, $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, I spent much effort trying to prove that that interval was $[1/4, 3/4]$. There were several minor clues pointing that way. Line (22) is one such; it will be noted that it assures a nonnegative payoff if $s \leqq 3/4$.

Thus, for this author at least, the solution of O'Toole's G^2 game presented a considerable challenge, even after the form of the solution had been guessed correctly, and it remained only to determine one number, δ, defining the admitted interval for s values.

These observations are worth mention because of their impact on the generalization I now propose.

Suppose that instead of being a 2×2 matrix, M is $m \times n$. Suppose that A and B have total resources of p and q respectively. In the first stage, each player covertly assigns mn nonnegative numbers to a matrix, such that A's total to p, and B's to q. They then play a matrix game on M, as before. Determine the class of all optimal strategies.

From the theory of this paper, it can be shown that the value is $p - q$, and each player has an optimal basic strategy. A has a balanced column m-set of strategies, in which each entry is p/n; and B similarly has a balanced row set. These first stage moves, combined with naive play in the second stage, attain the value $p - q$. But there is no obvious way to generalize the other results of this paper, and characterize the set of all optimal strategies. If the solver follows any line of thought similar to my own in attacking this problem, it appears that he will have to be lucky enough to guess the answer in advance *exactly*; "almost" won't be much help. This seems to suggest that a complete new approach may be needed to make much additional progress.

One small clue as to what such an approach might be can be gleaned from examination of the case in which the player had to depart from a naive strategy.

The naive strategy can be "explained" thus: When A invests all his resources in a single column at stage one, he is best off at stage two to try to capitalize on that choice by selecting that column. Any other choice would subject him to certain loss (or at best, zero payoff).

As we have seen, this reasoning is generally valid. But it sometimes fails. That occurs when A's stage one move puts unequal allocations, s and $1 - s$, in the two cells of the column. Then if B is fortunate enough to play a large portion of his assets in the cell where A has the lesser, it may still come out that A risks substantial loss by selecting that column. When this occurs, B will have only a small part of his resources left to assign to the other column. It may then be prudent for A to reject the naive strategy, with its risk of large loss, and choose the other column, with its certainty of much smaller loss. As we have shown, although A will suffer a loss on any such individual play of the G^2 game, his expectation over the balanced 4-set is always nonnegative.

(It is interesting that whether or not a naive policy is proper, a player can still always assure a zero return (i.e., no loss) over a basic or nonbasic 4-set by using pure strategies at the second stage. Does this generalize to the $m \times n$ case?)

This fragmentary rationale for including certain lines of play in the optimal strategies is hardly more than a vague hint. Is it enough to point any reader in the right direction, and enable him to fully characterize the optimal solution sets in the $m \times n$ extension of G^2? The author will be watching to see, with great interest.

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THE SERIES $\sum_{k=1}^{\infty} k^{-s}$, $s = 2, 3, 4, \cdots$, ONCE MORE

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Within the last few years a number of papers on the fascinating and mysterious (the distinction depends upon whether s is even or odd) series $\zeta(s) := \sum_{k=1}^{\infty} k^{-s}$, $s = 2, 3, 4, \cdots$ have been published. The solutions of the "Basler Problem" [**15**, p. 66], i.e., the problem of determining the sum $\sum_{k=1}^{\infty} k^{-2}$, one of vital importance to Euler and the Bernoulli brothers [**9**, p. 237, § 32. 136], have become simpler and

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$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6} \tag{1}$$

has become famous enough to be reproduced explicitly in the Encyclopaedia Britannica [**24**, p. 364].

Concerning this fundamental case $s = 2$ there are (in order of appearance) the papers [**22**], [**23**], [**10**], [**16**], [**6**], [**17**], [**4**], [**11**]. In the apparently unnoticed paper [**22**] formula (1) is derived in an exciting simple way using only de Moivre's identity; this result is represented anew one year later as "an elementary solution of a nonelementary mathematical problem" in the book of the same authors [**23**, p. 24, Ex. 145a; p. 131]. It is striking that the solutions given (independently) in [**6**] and [**11**] are identical with that of [**22**] (compare the remarks following the papers [**14**], [**11**]). The proofs of [**10**], [**16**], [**17**], [**4**] make use of well-known kernels of approximation theory such as those of de La Vallée Poussin, Fejér, and Dirichlet.

In the case of even powers $s = 2p$, $p = 1,2,3,\cdots$ one has the papers [**3**], [**20**], [**21**], [**18**], [**1**] and again [**22**], [**23**], with very different proofs being given. For example, in [**21**] a certain cotangent sum is estimated in two different ways using complex function theory. This yields the desired result

$$\zeta(2p)=(-1)^{p+1}\frac{(2\pi)^{2p}B_{2p}}{2(2p)!} \qquad (p=1,2,3,\cdots) \tag{2}$$

where B_{2p}, $p = 1,2,3,\cdots$, are the Bernoulli numbers of even order [**9**, p. 245].

This is also the method of [**18**], the sum being replaced by the even moments of Dirichlet's kernel, thus giving a straightforward generalization of [**16**], [**17**], [**4**]. The proof in [**1**] is very closely related to that in [**22**, 182–183] and [**23**, p. 24, Ex. 145b; p. 131–133] where the case $p = 2$ is considered explicitly and the obvious generalization to the remaining $p = 3,4,5,\cdots$ is indicated.

In the case of odd powers $s = 3,5,7,\cdots$ the literature is rather meager [**9**, p. 238, § 32. 136, 138], [**12**, p. 17]. Here one only has results of type [**7**], [**2**]

$$2\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^2}\left\{1+\frac{1}{2}+\cdots+\frac{1}{k}\right\},$$

or [**1**, p. 430]

$$\zeta(2p+1)=\left(\frac{\pi}{2}\right)^{2p+1}\lim_{m\to\infty}\frac{1}{m^{2p+1}}\sum_{k=1}^{m}\left(\cot\frac{k}{2m+1}\right)^{2p+1}$$

Of interest, though somewhat older, may be some other representations given in [**8**]. A very important contribution is [**19**] where $\zeta(3)$ and $\zeta(5)$ are expressed very simply in terms of the integral and the derivative, respectively, of a well-known transcendental function from elastostatics, thus giving a relation of these series to a physical object. The author gratefully remembers a profound discussion on the subject with

Professor F. G. Tricomi when he had given a lecture related to [19] at the Mathematical Colloquium of the Technological University of Aachen, July 11, 1969.

Apart from establishing this probably incomplete bibliography, the purpose of this note is to give a considerably simpler proof of the recursive formula (5) for $s = 2p$ obtained in [18] as well as to derive simultaneously formulae for the crucial odd case $s = 2p - 1$, $p \geqq 2$.

Instead of applying the kernel of Dirichlet we use Fejér's kernel

$$F_n(x) = \frac{1}{2(n+1)} \left[\frac{\sin (n+1)\frac{x}{2}}{\sin\frac{x}{2}} \right]^2 = \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kx. \tag{3}$$

By means of the identity

$$\int_0^{\pi} t^{2p} \cos kt \, dt = (-1)^k \pi^{2p+1} (2p)! \sum_{j=1}^{p} \frac{(-1)^{j+1}}{[2(p-j)+1]!(k\pi)^{2j}},$$

valid for any fixed $p = 1,2,3,\cdots$ [18], we derive for the even moments of (3)

$$\frac{1}{n+1} \int_0^{\pi} \frac{t^{2p}}{2} \left[\frac{\sin (n+1)\frac{t}{2}}{\sin\frac{t}{2}} \right]^2 dt$$

$$\begin{aligned} &= \pi^{2p+1} \left\{ \frac{1}{2(2p+1)} - (2p)! \sum_{k=1}^{n} (-1)^{k+1} \sum_{j=1}^{p} (-1)^{j+1} \frac{\pi^{-2j}}{[2(p-j)+1]!} \frac{1}{k^{2j}} \right\} \\ &\quad + \frac{1}{n+1} \left\{ \pi^{2p+1} (2p)! \sum_{k=1}^{n} (-1)^{k+1} \sum_{j=1}^{p} (-1)^{j+1} \frac{\pi^{-2j}}{[2(p-j)+1]!} \frac{1}{k^{2j-1}} \right\}, \end{aligned} \tag{4}$$

$$p = 1,2,3,\cdots.$$

For fixed p and $n \to \infty$, the integral on the left side of this equation and the bracketed factor of the second term on the right are obviously both bounded. Thus letting $n \to \infty$ we obtain

$$\sum_{k=1}^{\infty} (-1)^{k+1} \sum_{j=1}^{p} (-1)^{j+1} \frac{\pi^{-2j}}{[2(p-j)+1]!} \frac{1}{k^{2j}} = \frac{1}{2(2p+1)!}.$$

Extracting the term for $j = p$ and setting $A_{2p} = \Sigma_{k=1}^{\infty} (-1)^{k+1} k^{-2p}$, $p = 1,2,3,\cdots$, we get

$$A_{2p} = \pi^{2p} \left\{ \frac{(-1)^{p+1}}{2(2p+1)!} - \sum_{j=1}^{p-1} \frac{(-1)^{p+j}}{[2(p-j)+1]!\pi^{2j}} A_{2j} \right\}. \tag{5}$$

As an illustration, let us note the first three values of A_{2p}, namely

$$A_2 = \frac{1}{12}\pi^2, \; A_4 = \frac{7}{720}\pi^4, \; A_6 = \frac{31}{30240}\pi^6.$$

From the relations [9, p. 138]

$$\zeta(2p) = \frac{2^{2p}}{2^{2p}-2}A_{2p},\ U_{2p} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2p}} = \frac{2^{2p}-1}{2^{2p}-2}A_{2p} \tag{6}$$

one immediately derives the corresponding sums for these series [18].

Now, we start again from (4). As we noted before, the bracketed factor of the first term on the right converges to zero as $n \to \infty$. By the theorem on the remainder of an alternating series [9, p. 250] it follows that this factor is of order $O(n^{-2})$ for $n \to \infty$ since for each value of p the worst term occurring is

$$\frac{\pi^2}{12} - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^2} = \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k^2} = O(n^{-2}).$$

On the other hand, since $2\sin^2 x = 1 - \cos 2x$, the integral in (4) may be split up as

$$\int_0^{\pi} \frac{t^{2p}}{\sin^2 \frac{t}{2}} \sin^2(n+1)\frac{t}{2} dt = 2^{2p} \int_0^{\pi/2} \frac{t^{2p}}{\sin^2 t} dt - \frac{1}{4}\int_{-\pi}^{\pi} \frac{t^{2p}}{\sin^2 \frac{t}{2}} \cos(n+1)t\, dt$$

where the latter integral tends to zero for $n \to \infty$ by the lemma of Riemann-Lebesgue. Combining these results after multiplying both sides of (4) by $n+1$, we arrive at the limit

$$\sum_{j=1}^{p} \frac{(-1)^{j+1}}{[2(p-j)+1]!\pi^{2j}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2j-1}} = \frac{2^{2p-1}}{\pi^{2p+1}(2p)!} \int_0^{\pi/2} \frac{t^{2p}}{\sin^2 t} dt.$$

Denoting the even moments of the singular ($t = 0$) function $\left(\sin\frac{t}{2}\right)^{-2}$ by

$$M_{2p} = 2^{2p+1} \int_0^{\pi/2} \frac{t^{2p}}{\sin^2 t} dt = \int_0^{\pi} \frac{t^{2p}}{\sin^2 \frac{t}{2}} dt \quad (p = 1, 2, 3, \cdots), \tag{7}$$

the final recurrence formula reads

$$A_{2p-1} = \frac{(-1)^{p+1}}{4\pi(2p)!} M_{2p} - \sum_{j=1}^{p-1} (-1)^{p+j} \frac{\pi^{2(p-j)}}{[2(p-j)+1]!} A_{2j-1}. \tag{8}$$

Thus the problem of determining the numerical values of A_{2p-1} and

$$\zeta(2p-1) = \frac{2^{2p}}{2^{2p}-4}A_{2p-1},\ U_{2p-1} = \frac{2^{2p}-2}{2^{2p}-4}A_{2p-1} \qquad (p > 1)$$

is equivalent to the problem of evaluating the moments (7). This is easily carried out in case $p = 1$, namely [5, II, p. 123], [13, p. 166]

$$M_2 = 8\int_0^{\pi/2} \frac{t^2}{\sin^2 t} dt = 16 \int_0^{\pi/2} x \cot x dx = 8\pi \log 2,$$

thus yielding the familiar result for Leibniz' series

$$A_1 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2.$$

But for the remaining $p = 2,3,4,\cdots$ one only has as an illustration of (8)

$$A_3 = \frac{-1}{96\pi} M_4 + \frac{\pi^2}{6} A_1 .$$

Thus

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} = \frac{\pi^2}{6} \log 2 - \frac{1}{3\pi} \int_0^{\pi/2} \frac{t^4}{\sin^2 t}\, dt$$

and

$$A_5 = \frac{7\pi^3}{2880} M_2 - \frac{\pi}{576} M_4 + \frac{1}{2880\pi} M_6$$

or, explicitly,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5} = \frac{7\pi^3}{360} \int_0^{\pi/2} \frac{t^2}{\sin^2 t} dt - \frac{\pi}{18} \int_0^{\pi/2} \frac{t^4}{\sin^2 t} dt + \frac{2}{45\pi} \int_0^{\pi/2} \frac{t^6}{\sin^2 t}\, dt.$$

In general the integrals (7) cannot be evaluated in terms of elementary functions but one has [5, I, p. 130 (8c, 13d)]

$$\int_0^{\pi/2} \frac{t^{2p}}{\sin^2 t}\, dt = 2p \int_0^{\pi/2} x^{2p-1} \cot x \, dx = 2p \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{[2(p+k)-1](2k)!} \left(\frac{\pi}{2}\right)^{2(p+k)-1},$$

an infinite series involving again the Bernoulli numbers familar from (2).

It is ironical that the sums of the alternating version of U_{2p-1}, namely

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{2p-1}},$$

are well known, see [9, p. 239]!

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MULTIPLE SUBDIVISIONS OF THE PLANE

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It is known that the plane cannot be tiled by a lattice arrangement of translates of a convex polygon having more than six sides. A lattice multiple tiling of the plane by octagons is shown here, where every point of the plane (except those points lying on the boundary of an octagon) lies in exactly five octagons. A lattice multiple tiling by hexagons is shown which is irreducible in the sense that it is not possible to obtain from this configuration a simple subdivision by the removal of hexagons.

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1. Preliminaries. Let M be an n-dimensional convex body (compact, convex set with interior points). Minkowski [**6**] proved that if there is a lattice arrangement of translates of M which forms a simple subdivision of E^n (i.e., almost every point of E^n is contained in exactly one translate of M) then M must be a polytope with at most $2(2^n - 1)$ faces. For $n = 2, 3, 4$ polytopes are all known for which a subdivision of E^n exists for such a lattice arrangement of bodies (see Alexandrov [**1**], Coxeter [**2**], Groemer [**3**], and the references cited in those papers).

If the requirement of a lattice is dropped so that we have a simple subdivision of E^n by translates of a convex body M, Hlawka [**5**] proved that M must be a polytope with at most $3^n - 1$ faces. This result was generalized by Groemer [**3**] who showed that if there exists a simple subdivision of E^n by bodies which are homothetic to M, then each is a centrally symmetric polytope with at most $3^n - 3$ faces and each face is also centrally symmetric.

2. k-subdivisions in E^n. Suppose now we have a k-subdivision of E^n (a.e. point of E^n is contained in exactly k bodies) by translates of M. Obviously, one way to obtain such is to take a simple ($k = 1$) subdivision of E^n and overlap E^n with k such simple subdivisions. Is this the only way to form a k-subdivison ($k > 1$) or are there k-subdivisions which are irreducible in the sense indicated? Hajos [**4**] proved that in E^n ($n > 3$) there exists an integer k (which depends on n) for which there is a lattice k-subdivision of E^n by cubes, which cannot be reduced to a simple subdivision of E^n by the removal of certain ones of these cubes. He also proved that such a result does not hold for $n \leqq 3$.

3. Examples in the plane. We pose the following questions for the case $n = 2$:

(1) Does there exist a convex polygon M with more than $3^2 - 3 = 6$ sides and an integer $k > 1$ for which a k-subdivision of E^2 actually exists?

(2) Does there exist a lattice k-subdivision of E^2 consisting of translates of a convex polygon having more than $2(2^2 - 1) = 6$ sides?

(3) If there exists a k-subdivision ($k > 1$) of E^2 which is generated by a convex polygon M, will M always generate a simple subdivision of E^2?

(4) Given a lattice k-subdivision of E^2 which is generated by a convex polygon M, can a simple subdivision always be obtained by the removal of certain ones of these bodies?

(5) Given a k-subdivision ($k > 1$) of E^2 which is generated by a convex polygon M having at most six sides, can a simple subdivision be obtained by the removal of certain ones of these bodies?

If M is not a convex body, but an arbitrary compact connected set, these questions are easily answered. A nonconvex body is shown in Figure 1 for which there exists a 3-subdivision of E^2 by translates of this body but there does not exist a simple subdivision of E^2 by translates of this body.

The 3-subdivision of E^2 is obtained by translating the figure in (a) in such a way as to form a "band" as shown in (b) in which alternate squares are covered exactly once and twice. It is clear how to translate (b) and superimpose the resulting "band"

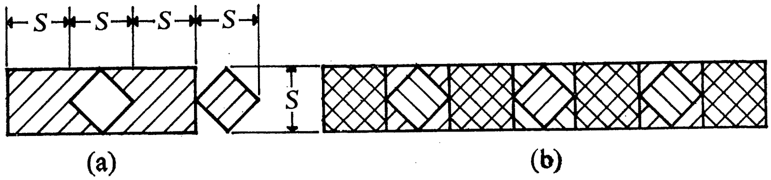

FIG. 1

on (b), so that the resulting "band" is covered 3 times. Since the plane can be covered by such "bands," we obtain a 3-subdivision of E^2.

The first four of these questions can all be answered by exhibiting a convex polygon M, with more than six sides, which generates a lattice k-subdivision ($k > 1$) of E^2. Questions (1) and (2) will then be answered affirmatively, and questions (3) and (4) negatively. Figure 2 shows an octagon with center at the origin and the lattice of the midpoints of the octagons which form a lattice 5-subdivision, by translates, of E^2. It is known (as mentioned in Section 1) that there does not exist a subdivision of E^2 by octagons.

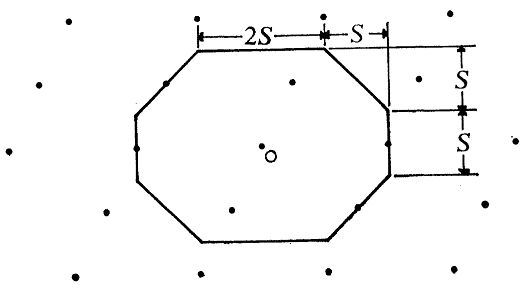

FIG. 2

The 5-subdivison of E^2 is obtained by considering four "layers" similar to that shown in Figure 3. Almost every point of the vertically hatched region is contained in

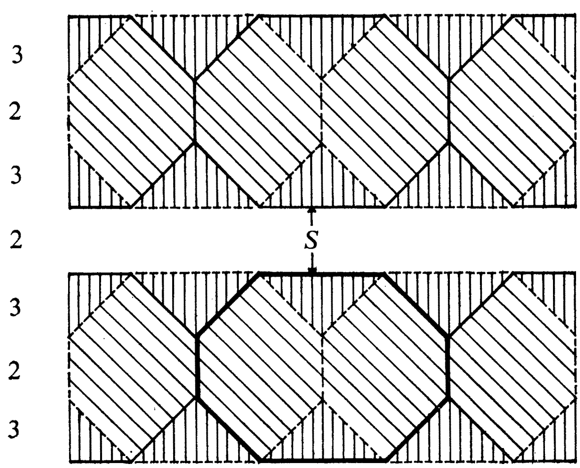

FIG. 3

one of the octagons in this "layer." Almost every point of the diagonally hatched region is contained in two of the octagons in this "layer." The points in the "open" region are not covered by any octagon from this "layer." By suitably translating this "layer" and superimposing the resulting "layers" on the former one, we obtain alternating "bands" covered twice and thrice as indicated by the numerals in Figure 3. It is then clear how one can obtain a 5-subdivision from this resulting configuration.

Examples have also been constructed of lattice k-subdivisions ($k > 1$) which are formed by translates of a decagon and a dodecagon. (In the case of the decagon, $k = 6$ and for the dodecagon $k = 35$.)

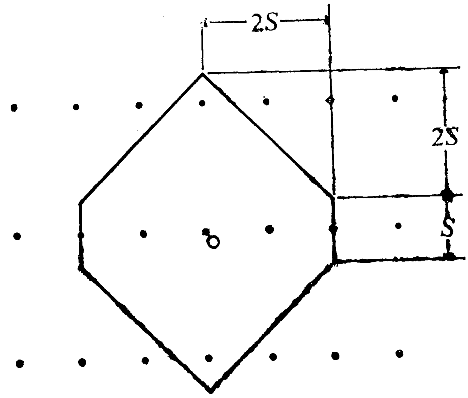

FIG 4

In order to answer the last question which was posed above, Figure 4 shows a a hexagon which generates a 6-subdivision, by translates of E^2 for which no simple subdivision can be obtained by removing bodies from this 6-subdivision. In order to have a simple subdivision, it is necessary that each boundary point of a body in the subdivision be covered at least twice; once by bodies which lie on each side of the line containing the boundary on which the point lies (see Groemer [3], Lemma 3). It may be seen from Figure 5 that this requirement cannot be met by removing sets from the 6-subdivision.

The 6-subdivision of E^2 is obtained by considering four "layers" similar to that shown in Figure 5 (a). Almost every point of the diagonally hatched region is contained in two of the hexagons in this "layer," and almost every remaining point is contained in exactly one hexagon of this "layer." Figures 5(a), (b), (c), (d), show that hexagon which is centered at the origin in relation to each of the four "layers" which form this 6-subdivision.

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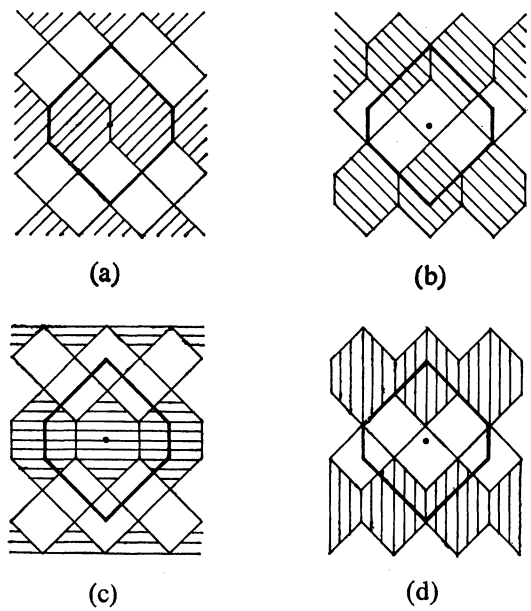

FIG. 5

4. G. Hajos, Subdividing multidimensional space with latticed cubes, (Hungarian), Mat. Lapok, 45 (1938) 171–190.

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THE PERIODS OF A PERIODIC FUNCTION

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Given a function f with domain R^1 and arbitrary range, we call, as usual, the positive real number p a period of f if $f(x+p)=f(x)$ for all $x \in R^1$. In discussions of periodic real functions and their Fourier series it is usually tacitly assumed that every nonconstant function under consideration has a fundamental period of which all other periods are integral multiples. Justification for this assumption exists in the literature, but in an inconvenient form. The basic result is that if a Lebesgue measurable function has arbitrarily small periods, then it is almost everywhere constant. This theorem is stated in [2] (p. 234), but the proof, which also involves [1], is somewhat inaccessible and depends on a specialized and none too easily proved criterion for nonmeasurability. It is the principal object of this note to supply a simple proof which uses only standard theorems from measure theory. The existence of a fundamental period of a not almost everywhere constant periodic function is a simple

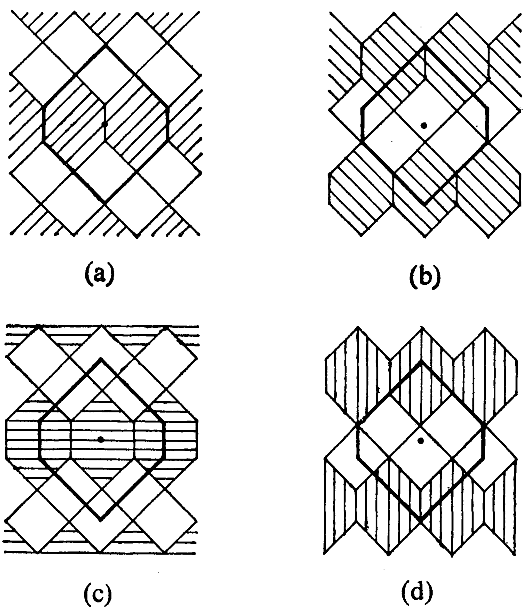

FIG. 5

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LEMMA 1. *Let E be a (Lebesgue) measurable subset of R^1 with $0 < m(E) < \infty$. Also let $f: E \to \bar{R}$ (the extended real number system) be a measurable function which is not almost everywhere constant. Then there exists a subset A of $\bar{R}$ such that $f^{-1}(A)$ is measurable and*

$$0 < m(f^{-1}(A)) < m(E). \tag{1}$$

Proof. (i) If there exists $y \in \bar{R}$ such that $m(f^{-1}(\{y\})) > 0$, then, since f is not a.e. constant, $A = \{y\}$ satisfies (1).

(ii) Suppose that $m(f^{-1}(\{y\})) = 0$ for all $y \in \bar{R}$. For $n = 1, 2, \cdots$ and $r = 0, \pm 1, \pm 2, \cdots$, put

$$A_{nr} = \left[\frac{r}{2^n}, \frac{r+1}{2^n}\right].$$

Then, for each n,

$$\sum_{r=-\infty}^{\infty} m(f^{-1}(A_{nr})) = m(E).$$

We show that there exist n and r such that $A = A_{nr}$ satisfies (1). For assume that this is false. Then for each n there exists an integer $\rho(n)$ such that, if $B_n = A_{n\rho(n)}$,

$$m(f^{-1}(B_n)) = m(E). \tag{2}$$

Since $m(f^{-1}(A_{nr})) = 0$ when $r \neq \rho(n)$, $B_{n+1} \subseteq B_n$. It follows that

$$\bigcap_{n=1}^{\infty} B_n = \{b\}$$

for some real number b. Also $f^{-1}(B_{n+1}) \subseteq f^{-1}(B_n)$ and

$$\bigcap_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}(\{b\}),$$

so that

$$m(f^{-1}(B_n)) \to m(f^{-1}(\{b\})).$$

Hence, by (2), $m(f^{-1}(\{b\})) = m(E) > 0$ and this contradicts our original assumption. Thus the lemma is proved.

LEMMA 2. *Let A be a subset of $\bar{R}$ such that $A \cap R^1$ is measurable. If the measurable function $f: R^1 \to \bar{R}$ is periodic, then*

$$\frac{m(f^{-1}(A) \cap [a, a+p])}{p}$$

has the same value for all $a \in R^1$ and all periods p of f.

Proof. We first note that our hypotheses ensure the measurability of the set

$f^{-1}(A)$. Next, if p_0 is any period of f, then

$$f^{-1}(A) \cap [np_0, (n+1)p_0]$$

has the same measure for $n = 0, \pm 1, \pm 2, \cdots$. It follows easily that

$$\frac{1}{X} m(f^{-1}(A) \cap [0, X]) \to \frac{1}{p_0} m(f^{-1}(A) \cap [0, p_0]) \text{ as } X \to \infty.$$

This means that $m(f^{-1}(A) \cap [0,p])/p$ has the same value for all periods p of f; and the statement of the lemma is an almost immediate consequence.

THEOREM 1. *If the measurable function $f: R^1 \to \bar{R}$ has arbitrarily small periods, then f is almost everywhere constant.*

Proof. Suppose that f is not almost everywhere constant. If p is a period of f, then f is not almost everywhere constant on $[0,p]$ and, by Lemma 1, there exists a subset A of $\bar{R}$ such that

$$(3) \qquad 0 < m(f^{-1}(A) \cap [0,p]) < p.$$

By Lemma 2, $m(f^{-1}(A) \cap [a, a+p])/p$ has the same value λ for all $a \in R^1$ and all periods p of f; and (3) shows that $0 < \lambda < 1$.

Since $m(f^{-1}(A)) > 0$, $f^{-1}(A)$ has a point of density α; thus

$$m(f^{-1}(A) \cap [\alpha, \alpha + \delta])/\delta \to 1$$

as $\delta \to 0$. But this is impossible as the left hand side has the value $\lambda < 1$ when δ is a period of f, and there are arbitrarily small periods. So f is a.e. constant.

COROLLARY. *If the continuous function $f: R^1 \to R^1$ has arbitrarily small periods, then f is constant.*

A direct proof of this result is given in [**3**] on p. 1.

Theorem 1 has an n-dimensional analogue: if the measurable function $f: R^n \to \bar{R}$ has arbitrarily small periods in each variable, then it is constant almost everywhere. For Lemma 1 holds, in fact, with an arbitrary measure space $(X, \mathscr{S}, \mu)$ in place of the Lebesgue measure space for which it has been stated, and the adaptations required in Lemma 2 and in the proof of Theorem 1 are mainly notational.

An illustration of Theorem 1 is provided by the function f on R^1 which has the value 1 at rational points and 0 at irrational points. Every positive rational number is a period of f. It is also worth noting that the condition of measurability in Theorem 1 is essential, for there exists a nonmeasurable function $g: R^1 \to R^1$ which has arbitrarily small periods (and is not a.e. constant).

To construct such a function we first take a partition of $[0,1)$ into nonmeasurable sets $S_1, S_2, \cdots$ with the following property: if $x \in S_n$ and $y \in [0,1)$, then $y \in S_n$ if and only if $y - x$ is irrational. (See, for instance, [**4**], pp. 142–144.) For each n, let

$$T_n = \bigcup_{k=-\infty}^{\infty} (S_n + k).$$

Then, when $x \in T_n$, $x + w \in T_n$ if and only if w is irrational.

We now define the function g on R^1 by the equation

$$g(x) = n \text{ when } x \in T_n \ (n = 1, 2, \cdots).$$

Clearly g is not measurable (and so cannot be a.e. constant). Also every positive irrational number is a period of g.

With the help of Theorem 1 it is easy to show that, if the periodic function $f: R^1 \to \bar{R}$ does not have arbitrarily small periods, then f has a smallest period ω and every period of f is an integral multiple of ω.

For let ω be the infimum of the set of periods of f. By hypothesis, $\omega > 0$. To show that ω is itself a period assume the contrary. Then there exists a decreasing sequence (p_n) of periods with limit ω. But $p_n - p_{n+1}$ is a period of f and $p_n - p_{n+1} \to 0$ as $n \to \infty$. This is impossible since f does not have arbitrarily small periods.

Now let p be any period of f. Then $p \geqq \omega$ and, for some integer n, $n\omega \leqq p < (n+1)\omega$. If $p \neq n\omega$, then $0 < p - n\omega < \omega$ and so $p - n\omega$ is a period smaller than ω. The contradiction implies that $p = n\omega$.

We finally consider the Fourier series corresponding to the different periods of a periodic function. It is convenient to begin with a definition.

Let

$$\sum_{m=0}^{\infty} a_m, \quad \sum_{n=0}^{\infty} b_n \tag{4}$$

be any two complex series, and denote by

$$(a_{\mu(0)}, a_{\mu(1)}, \cdots), \quad (b_{\nu(0)}, b_{\nu(1)}, \cdots) \tag{5}$$

the subsequences consisting of all nonzero terms of (a_m) and (b_n) respectively. We call the series (4) *similar* if the sequences (5) are identical.

Evidently similarity is an equivalence relation. Similar series have the same convergence properties, but their behavior with regard to summability may differ.

THEOREM 2. *If the function $f: R^1 \to \bar{R}$ is periodic and Lebesgue summable, then the Fourier series corresponding to any two periods of f are similar.*

Proof. Let the Fourier series of f corresponding to the period p be

$$c_0(p) + \sum_{n=1}^{\infty} (c_n(p)e^{i2n\pi x/p} + c_{-n}(p)e^{-i2n\pi x/p}).$$

(i) If f has arbitrarily small periods, then there is a real number κ such that $f(x) = \kappa$ a.e. Therefore, for all periods p, $c_0(p) = \kappa$ and $c_n(p) = 0$ when $n \neq 0$.

(ii) If f does not have arbitrarily small periods, then f has a smallest period ω and any period p of f is of the form $p = k\omega$, where k is a natural number. We shall show that

$$c_r(k\omega) = \begin{cases} c_n(\omega) & \text{if } r = kn \ (n = 0, \pm 1, \pm 2, \cdots), & (6) \\ 0 & \text{if } k \text{ does not divide } r. & (7) \end{cases}$$

For any integer r,

$$c_r(k\omega) = \frac{1}{k\omega}\int_0^{k\omega} f(x)e^{-i2r\pi x/(k\omega)}dx$$

$$= \frac{1}{k\omega}\sum_{l=0}^{k-1}\int_{l\omega}^{(l+1)\omega} f(x)e^{-i2r\pi x/(k\omega)}dx$$

and, since f has period ω,

$$c_r(k\omega) = \frac{1}{k\omega}\sum_{l=0}^{k-1}\int_0^{\omega} f(t)e^{-i2r\pi(t+l\omega)/(k\omega)}dt$$

$$= \frac{1}{k\omega}\left(\sum_{l=0}^{k-1} e^{-i2r\pi l/k}\right)\int_0^{\omega} f(t)e^{-i2r\pi t/(k\omega)}dt.$$

The sum in brackets is k when $r = kn$ ($n = 0, \pm 1, \pm 2, \cdots$) and is 0 when k does not divide r. This proves (6) and (7) and also the theorem.

Although Theorem 2 has been stated for Lebesgue integrable functions, it holds equally for functions integrable in more general senses, such as Denjoy's.

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SOME CONGRUENCES OF THE FIBONACCI NUMBERS MODULO A PRIME P

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MARJORIE BICKNELL, A. C. Wilcox High School, Santa Clara, California

1. Introduction. In [1], Leonard posed several conjectures regarding congruences satisfied by the Fibonacci numbers and the Lucas numbers modulo a prime p. The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$, defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_n + F_{n+1},$$

has been explored in numerous sources, including [2], [3], [4] and [5]. The Lucas numbers 1, 3, 4, 7, 11, 18, $\cdots$, are defined recursively in a similar manner as

$$L_1 = 1, L_2 = 3, \quad L_{n+2} = L_n + L_{n+1}.$$

The conjectures of Leonard [1] considered in this paper follow.

For any integer r,

$$c_r(k\omega) = \frac{1}{k\omega}\int_0^{k\omega} f(x)e^{-i2r\pi x/(k\omega)}dx$$

$$= \frac{1}{k\omega}\sum_{l=0}^{k-1}\int_{l\omega}^{(l+1)\omega} f(x)e^{-i2r\pi x/(k\omega)}dx$$

and, since f has period ω,

$$c_r(k\omega) = \frac{1}{k\omega}\sum_{l=0}^{k-1}\int_0^{\omega} f(t)e^{-i2r\pi(t+l\omega)/(k\omega)}dt$$

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$$L_1 = 1, L_2 = 3, \quad L_{n+2} = L_n + L_{n+1}.$$

The conjectures of Leonard [1] considered in this paper follow.

Conjecture 6. $F_p \equiv \pm 1 \pmod p$ if and only if p is a prime, $p \neq 5$.

The *only if* portion of Conjecture 6 is false by virtue of $F_4 \equiv -1 \pmod 4$ and $F_{22} \equiv +1 \pmod{22}$. However, we will establish that $F_p \equiv \pm 1 \pmod p$ if p is a prime, $p \neq 5$.

Conjecture 7. $L_p \equiv 1 \pmod p$ if and only if p is a prime.

Again, while the *if* portion will be proved to be correct, the *only if* portion of the conjecture is false, as illustrated by the counterexample,

$$L_{705} \equiv 1 \pmod{705},$$

(where, of course, 705 is not prime) as found by Joseph Greener of Lawrence Livermore Laboratory on a CDC 6600. (The next counterexamples occur for L_{2465}, L_{2737}, L_{3745}, and L_{4181}.)

This counterexample, while hard to find, is not difficult to check. Naturally, one does not attack L_{705} directly. Since $705 = 3 \cdot 5 \cdot 47$, one finds that the period of 3 is 8, 5 is 4, and 47 is 32 for the Lucas sequence as given in [**4**]. Thus, $L_{705} \equiv L_1$ (mod 3) since $705 = 8 \cdot 88 + 1$; $L_{705} \equiv L_1$ (mod 5) since $L_{705} = 4 \cdot 176 + 1$; and $L_{705} \equiv L_1$ (mod 47) since $705 = 32 \cdot 22 + 1$. Therefore, since 3, 5, and 47 are primes, $L_{705} \equiv 1$ (mod 705).

Conjecture 8. $F_{pk} \equiv \pm F_k \pmod p$ if p is a prime and $(k, p) = 1$, $p \neq 5$.

Conjecture 9. $L_{pk} \equiv \pm L_k \pmod p$ if p is a prime and $(k, p) = 1$.

Both Conjectures 8 and 9 are true. Improved versions are proved in what follows.

2. Proofs of modified conjectures.

THEOREM 6*. *For p a prime, $F_p \equiv 1 \pmod p$ if $p = 5k \pm 1$ and $F_p \equiv -1 \pmod p$ if $p = 5k \pm 2$.*

The proof is contained in the proof of Theorem 180 in [**5**] a part of which we state for use in Theorem 7*:

For p a prime, $F_{p-1} \equiv 0 \pmod p$ if $p = 5k \pm 1$ and $F_{p+1} \equiv 0 \pmod p$ if $p = 5k \pm 2$.

From the identity $F_{p-1}F_{p+1} - F_p^2 = (-1)^p$, it follows from Theorem 180 that $F_p^2 \equiv 1 \pmod p$, so that we have $F_p \equiv \pm 1 \pmod p$, p a prime, $p \neq 5$. Further, from [**5**],

$$F_p \equiv \begin{cases} +1 \pmod p, & p = 5m \pm 1 \\ -1 \pmod p, & p = 5m \pm 2. \end{cases}$$

Therefore, we have proved Theorem 6*. Since $F_5 = 5 \equiv 0$ (mod 5), the restriction $p \neq 5$ is necessary.

THEOREM 7*. *If p is a prime, then $L_p \equiv 1 \pmod p$.*

Proof. Since $L_5 = 11 \equiv 1 \pmod 5$, we use Theorem 180 and Theorem 6* and need consider only primes of the two types considered there. We further use $L_p = F_{p+1} + F_{p-1}$ and $F_{p+1} = F_p + F_{p-1}$. If $p = 5k \pm 1$, then $L_p \equiv F_{p+1} = F_p + F_{p-1} \equiv 1 \pmod p$. If $p = 5k \pm 2$, then $L_p \equiv F_{p-1} = F_{p+1} - F_p \equiv 0 - (-1) \equiv 1 \pmod p$.

THEOREM 8*. *If $p \neq 5$ is a prime, then $F_{kp} \equiv \pm F_k \pmod p$, where the minus sign occurs for p of the form $5k \pm 2$.*

Proof. From [6], one obtains

$$F_{kn} = F_n \sum_{j=0}^{[k/2]} \binom{k-1-j}{j} L_n^{k-2j-1}, \qquad n \text{ odd}. \tag{2.1}$$

Since $L_p \equiv 1 \pmod p$ when p is a prime and since, taking $n = 1$ in (2.1),

$$F_k = \sum_{j=0}^{[k/2]} \binom{k-1-j}{j},$$

then (2.1) yields

$$F_{kp} \equiv F_p F_k \equiv \begin{cases} + F_k \pmod p & \text{if } p = 5m \pm 1, \\ - F_k \pmod p & \text{if } p = 5m \pm 2, \end{cases}$$

where p is an odd prime, $p \neq 5$. For the prime 2, consider

$$F_{2k} \equiv L_k F_k \pmod 2.$$

Clearly

$$F_n \equiv 1, 1, 0, \cdots \pmod 2 \qquad (n = 1, 2, 3, \cdots),$$

$$L_n \equiv 1, 1, 0, \cdots \pmod 2 \qquad (n = 1, 2, 3, \cdots).$$

Thus $F_{2k} \equiv F_k \pmod 2$, $k \not\equiv 0 \pmod 3$, since then $F_k \equiv L_k \equiv 1 \pmod 2$. If $k \equiv 0 \pmod 3$, then $F_{2k} \equiv F_k \equiv 0 \pmod 2$, so $F_{2k} \equiv F_2 F_k \pmod 2$, which concludes the proof of Theorem 8* for any prime p, $p \neq 5$. Notice that the original restriction $(p, k) = 1$ has been removed for $p = 2$. It does not enter otherwise.

From (2.1) and Theorem 7*, $F_{5k} \equiv F_5 F_k \pmod 5$ from which it follows that $F_{5k} \equiv 0 \pmod 5$, for all k. This shows that Theorem 8* is false when $p = 5$. Examples such as $F_{16} \equiv F_4 \pmod 4$, $F_{50} \equiv F_5 \pmod{10}$, and $F_{27} \equiv F_3 \pmod 9$ show that the converse of Theorem 8* is false.

THEOREM 9*. $L_{pk} \equiv L_k \pmod p$ *if p is a prime.*

Proof. In [7] it is shown that

$$L_n^{m+1} = L_{(m+1)n} + \sum_{i=1}^{[(m+1)/2]} C_{m+1,i}(-1)^{ni+i-1} L_n^{m+1-2i}$$

where each $C_{k,m}$ is the sum of two binomial coefficients,

$$C_{k,m} = \binom{k-m-1}{m-1} + \binom{k-m}{m},$$

$$C_{k,0} = 1, \quad C_{m,j} = C_{m-1,j} + C_{m-2,j-1}, \quad 1 \leqq j \leqq [m/2], \quad m \geqq 2,$$

from which one can easily derive

$$L_{(m+1)n} = \sum_{i=0}^{[(m+1)/2]} C_{m+1,i}(-1)^{(n+1)i} L_n^{m+1-2i}.$$

Thus, if n is an odd prime p, let $k = m+1$, then

$$L_{kp} = \sum_{i=0}^{[k/2]} C_{k,i} L_p^{k-2i}$$

and

$$L_k = \sum_{i=0}^{[k/2]} C_{k,i}.$$

Therefore, since $L_p \equiv 1 \pmod p$, p an odd prime, $L_{kp} \equiv L_k \pmod p$.

Again we take the case $p = 2$ separately. Observe that

$$L_k \equiv 1, 1, 0, \cdots \pmod 2, \quad (k = 1, 2, 3, \cdots),$$

i.e., $L_k \equiv 1 \pmod 2$ exactly when $k \not\equiv 0 \pmod 3$. If $k \not\equiv 0 \pmod 3$, then $2k \not\equiv 0 \pmod 3$ and hence $L_{2k} \equiv L_k \equiv 1 \pmod 2$, while $L_{2k} \equiv L_k \equiv 0 \pmod 2$ if $k \equiv 0 \pmod 3$. Thus, $L_{2k} \equiv L_k \pmod 2$ without restrictions on k when $p = 2$. This concludes the proof of Theorem 9* for all primes p, removing the restriction given in the original conjecture.

3. Some further results. Conjectures 8 and 9 can be extended in several interesting ways. It is not difficult to prove the following results by mathematical induction; the proofs are omitted for brevity.

(3.1) $$L_{2^s n} \equiv L_n^{2^s} \pmod 2, \qquad (s = 0, 1, 2, 3, \cdots)$$

(3.2) $$L_{2^k} \equiv -1 \pmod{2^k},$$

(3.3) $$F_{2^k} \equiv (-1)^{k+1} \pmod 4, \; k \geqq 1,$$

(3.4) $$F_{p^s k} \equiv \pm F_k \pmod p, \; p \neq 5, \; (s = 0, 1, 2, 3, \cdots),$$

(3.5) $$L_{p^s k} \equiv L_k \pmod p, \qquad (s = 0, 1, 2, 3, \cdots),$$

(3.6) $$L_{p^k+1} L_{p^k-1} \equiv 6 \pmod p.$$

Acknowledgement: The conjectures of Leonard [1] overlap somewhat some similar conjectures of J. A. H. Hunter given in a private communication in 1966. Congruence (3.6) is due to Vinh Phat in a private communication.

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ARE THE TWIN CIRCLES OF ARCHIMEDES REALLY TWINS?

LEON BANKOFF, Los Angeles, California

If a point C is selected anywhere on a line segment AB and semicircles are constructed on the same side of AC, CB and AB, the region enclosed by the three semicircular arcs is known as the Shoemaker's Knife or the Arbelos. Let the half-chord CD perpendicular to AB divide the arbelos into two mixtilinear triangles, each bounded by a straight line and two curved lines. It has been known for more than 2000 years that the circles inscribed in the regions ACD and BCD are always equal. The proof of this theorem appears as Proposition 5 of the *Book of Lemmas* of Archimedes, and it seems appropriate to call these inscribed circles "The Twin Circles of Archimedes."

Now reflect the Shoemaker's Knife in the diameters AB, AC and CB, omitting the line CD and the twin circles, but instead let the reflected figure contain a circle tangent to the arcs AB, AC and CB at the points P, Q, R. The calculation of the diameter of the circle PQR is considered in Proposition 6 of the Book of Lemmas. Although the properties of the Shoemaker's Knife have proliferated and expanded into the most fascinating ramifications during the passing centuries, it appears that one of the most striking properties of the arbelos has remained unnoticed until now. It can be shown that *the circle determined by the points C, Q, R is always equal to the corresponding pair of circles inscribed in the segmented arbelos*. The intimate association of these three circles suggests that the appellation "Twin Circles of Archimedes" is a misnomer. Actually we now have a triad instead of a dyad of equal circles to perplex and astonish the neophyte and the sophisticate alike (Figure 1).

Archimedes' proof of Proposition 5 is unquestionably a masterpiece of ingenuity and a splendid example of the techniques of his time. An interesting sidelight connected with his demonstration is that it contains the first recorded proof that the

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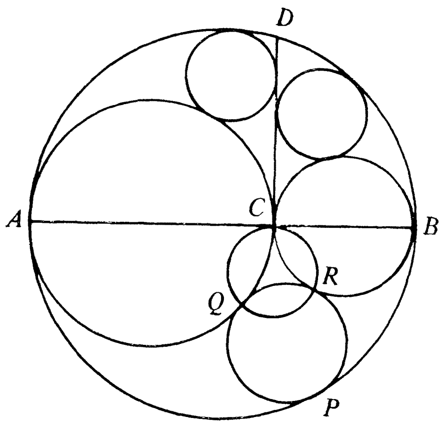

FIG 1.

altitudes of a triangle are concurrent—an item Euclid must certainly have known but somehow never got around to writing up. The very presence of this seemingly irrelevant theorem and the heavy reliance on the preliminary lemmas of the first three propositions lead us to conclude that the proof is a bit involved and cumbersome by present day standards. For more details the reader is referred to the Dover edition of Heath's *Works of Archimedes,* which contains the Book of Lemmas in its entirety.

For those who prefer the simplicity of an algebra that was unknown at the time of Archimedes, we offer a brief proof that requires nothing more complicated than high school geometry. Denote the centers of the semicircles AB, AC, CB by O, O_1, O_2, and let $AB = 2r$, $AC = 2r_1$ and $CB = 2r_2$. Call the centers of the inscribed circles W_1 and W_2, their radii p_1 and p_2, and let X_1, X_2 denote the projections of W_1, W_2 upon AB (Figure 2). We then have

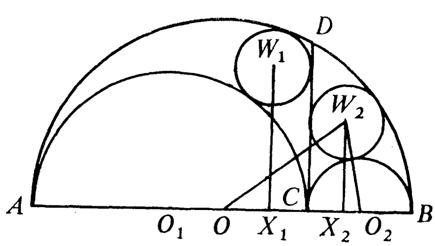

FIG. 2

$$OW_2^2 - OX_2^2 = O_2W_2^2 - O_2X_2^2$$

$$(r_1 + r_2 - p_2)^2 - (r_1 - r_2 + p_2)^2 = (p_2 + r_2)^2 - (r_2 - p_2)^2$$

$$2r_1(2r_2 - 2p_2) = 2r_2(2p_2)$$

and

$$p_2 = r_1 r_2 / r.$$

Similarly, starting with $OW_1^2 - O_1X_1^2 = O_1W_1^2 - O_1X_1^2$, we arrive at the result $p_1 = r_1r_2/r$, as before.

An equally simple proof stems from a little known theorem found in Casey's *Sequel to Euclid*, Dublin 1885, page 118: *If a variable circle touches two fixed circles, its radius has a constant ratio to the perpendicular from its center upon the radical axis.* Let the positions of the circles (W_1) and (O_2) be looked upon as those of a variable circle inscribed in the horn angle bounded by arcs AC and AB and equate p_1/r_2, the ratio of their radii, to the ratio of the distances of their centers to the tangent at A, namely the radical axis of (O_1) and (O). We find that $p_1/r_2 = (2r_1 - p_1)/(2r_1 + r_2)$, which is easily simplified to yield $p_1 = r_1r_2/(r_1 + r_2) = r_1r_2/r$. A similar procedure involving (O_1) and (W_2) in relation to the tangent at B yields the same result for p_2.

The calculation of the radius of the single circle PQR inscribed in the Shoemaker's Knife also lends itself to numerous methods of solution. The problem section of the November 1952 issue of this MAGAZINE contains seven different approaches, each one, of course, leading to the result that r_3, the radius of the circle (O_3) inscribed in the arbelos, is equal to $rr_1r_2/(r_1^2 + r_1r_2 + r_2^2)$.

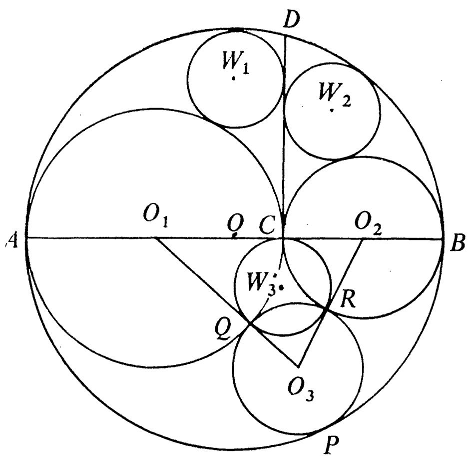

FIG. 3

The easiest way to determine the value of p_3, the radius of the circle (W_3) passing through the points C, Q, R, is to use a property made known by Pappus, namely, that the circumference of the circle (O_3) bisects the altitude to the base O_1O_2 of the triangle $O_1O_2O_3$ (Figure 3). We then combine the Heronian formula for the area of triangle $O_1O_2O_3$ with the "half the base times the altitude" formula as follows:

$$r_3(r_1 + r_2) = p_3(r_1 + r_2 + r_3) = \sqrt{r_1r_2r_3(r_1 + r_2 + r_3)}$$
$$= r_1r_2r_3(r_1 + r_2 + r_3)/p_3(r_1 + r_2 + r_3) = r_1r_2r_3/p_3,$$

with the result that $p_3 = r_1r_2/(r_1 + r_2) = r_1r_2/r$, thus proving that the *three* circles are equal.

Incidentally, the substitution of $p_3 = r_1r_2/r$ in the equation

$$r_3(r_1 + r_2) = p_3(r_1 + r_2 + r_3)$$

yields the result already noted:

$$r_3 = rr_1r_2/(r^2 - r_1r_2) = rr_1r_2/(r_1^2 + r_1r_2 + r_2^2).$$

As a bonus for the more advanced geometrician, we offer an example of how all the formulas just derived can be obtained in one fell swoop by a single inversion.

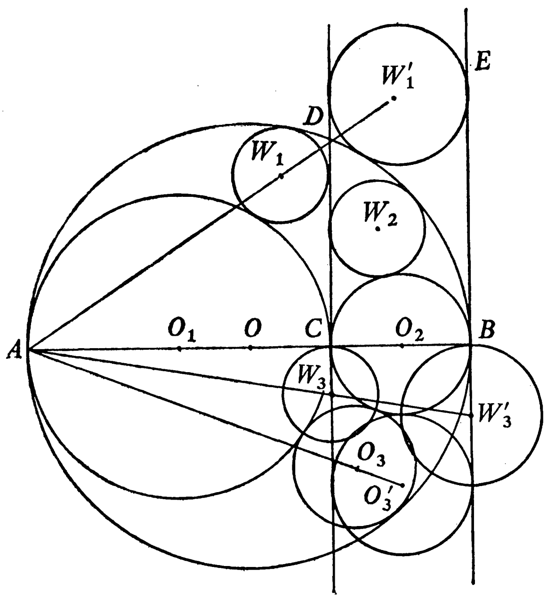

FIG. 4

Figure 4 represents the inversion centered at A with the power of inversion equal to AD^2, or $4rr_1$. The following transformations occur (with the previous notation retained):

The circle (O_1) becomes the line BE perpendicular to AB.

The circle (O) becomes the line CD extended.

The circles (O_2) and (W_2) are self-inverse.

The circle (W_1) becomes the circle (W_1') of radius p_1', tangent to lines CD and BE and touching circle (O) externally.

The circle (O_3) becomes the circle (O_3') of radius r_3', tangent to lines CD and BE and to the self-inverse circle (O_2).

The circle (W_3) becomes the circle (W_3') of radius p_3', tangent to AB at B and orthogonal to (O_2) and (O_3'), the inverses of (O_2) and (O_3).

The following conclusions are evident:

(1) Since A is the external center of similtude of (W_1), (W_1'), we have $p_1/p_1' = (2r_1 - p_1)/(2r_1 + r_2)$, from which we obtain $p_1 = r_1r_2/r$, with a similar result for p_2 in the inversion B, BD^2.

(2) In the triangles ACW_3 and ABW_3' we have $p_3/p_3' = AC/AB = r_1/r$ and it follows that $p_3 = r_1p_3'/r = r_1r_2/r$.

(3) The value of r_3, the radius of circle (O_3), is most conveniently found by using the relation $r_3/r_3' = AD^2/t^2$, where t is the length of the tangent from A to the circle (O_3'). It is easily seen that

$$t^2 = (AO_3')^2 - r_2^2 = (2r_1 + r_2)^2 + (2r_2)^2 - r_2^2 = 4(r_1^2 + r_1r_2 + r_2^2).$$

Then, since $r_3' = r_2$, we obtain

$$r_3 = r_2(2r_1)(2r)/4(r_1^2 + r_1r_2 + r_2^2) = rr_1r_2/(r_1^2 + r_1r_2 + r_2^2).$$

(4) It is clear from the figure that the distance from the center of (O_3') to AB is equal to the diameter of (O_3'), with a similar relation for the circle (O_3), thus verifying the Pappus property mentioned before.

Conclusion. The use of the reflected Shoemaker's Knife has been a convenient device for sidestepping complicated and undecipherable diagrams. The properties discussed here apply as well to superimposed congruent figures. Accordingly it would not be amiss to say that an invisible circle equal to the two inscribed circles has been lurking unnoticed in Archimedes' "twin circle" diagram. To answer the question posed in the title of this paper: "Are the twin circles of Archimedes really twins?", we can now state unequivocably, "No, they are merely two members of a set of triplets".

The third circle made its first printed appearance in my article entitled *A Mere Coincidence*, published in the Los Angeles Mathematics Newsletter dated November 1954. It was also discussed in my lectures at the California Conference for Teachers of Mathematics, held at the University of California at Los Angeles on July 14, 1954, at which time mimeographed notes were distributed to the audience. These lectures were later repeated at Pasadena City College, Occidental College, Pepperdine College and Pomona College.

AREA-DIAMETER RELATIONS FOR TWO-DIMENSIONAL LATTICES

P. R. SCOTT, University of Malaya

Let R be a convex region in the plane which contains no points of the integral lattice Γ. Let $A(R), P(R), d(R)$ respectively denote the area, perimeter and diameter of R. Bender [1] has shown that $A(R) \leqq \frac{1}{2}P(R)$. We establish a relation between the area and diameter of R.

THEOREM 1. *Let ϕ^* denote the unique solution of $\sin\phi = \pi/2 - \phi$, and set $\lambda = 2\sqrt{2}\sin(\frac{1}{2}\phi^*)$ ($\approx$ 1.144). Then $A(R) \leqq \lambda\, d(R)$, and this result is best possible.*

The following conclusions are evident:

(1) Since A is the external center of similtude of (W_1), (W_1'), we have $p_1/p_1' = (2r_1 - p_1)/(2r_1 + r_2)$, from which we obtain $p_1 = r_1r_2/r$, with a similar result for p_2 in the inversion B, BD^2.

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$$t^2 = (AO_3')^2 - r_2^2 = (2r_1 + r_2)^2 + (2r_2)^2 - r_2^2 = 4(r_1^2 + r_1r_2 + r_2^2).$$

Then, since $r_3' = r_2$, we obtain

$$r_3 = r_2(2r_1)(2r)/4(r_1^2 + r_1r_2 + r_2^2) = rr_1r_2/(r_1^2 + r_1r_2 + r_2^2).$$

(4) It is clear from the figure that the distance from the center of (O_3') to AB is equal to the diameter of (O_3'), with a similar relation for the circle (O_3), thus verifying the Pappus property mentioned before.

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THEOREM 1. *Let* ϕ^* *denote the unique solution of* $\sin\phi = \pi/2 - \phi$, *and set* $\lambda = 2\sqrt{2}\sin(\frac{1}{2}\phi^*)$ (≈ 1.144). *Then* $A(R) \leqq \lambda\, d(R)$, *and this result is best possible.*

The proof of the theorem can be greatly simplified using *Steiner symmetrization.* Let R be a given region and l a given line in the plane. We construct a new region R' which is symmetric about l as follows. Displace each chord AB of R which is perpendicular to l along the line of AB until its midpoint lies on l. Then R' is the union of these displaced chords. It is well known that symmetrization preserves convexity and areas, and does not increase diameters (see for example [2]).

LEMMA. *If R is a convex region containing no points of Γ, then there exists another convex region R^* containing no points of Γ such that*

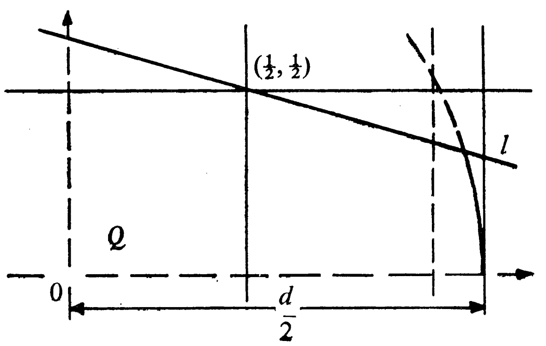
(a) *R^* is symmetric about the lines $x = \frac{1}{2}$, $y = \frac{1}{2}$*

(b) $A(R^*) = A(R)$, $d(R^*) \leqq d(R)$.

Proof. Let R' be the region obtained from R by symmetrization about the line $x = \frac{1}{2}$. Then R' is convex and $A(R') = A(R)$, $d(R') \leqq d(R)$. Further, R' contains no point of Γ. For if R' contained a point of Γ, then for some integer k the line $y = k$ would intercept R' (and so R) in a line segment of length greater than 1. This would imply that R contained a point of Γ, contradicting the hypothesis of the lemma.

A similar argument shows that if we now symmetrize R' about the line $y = \frac{1}{2}$, we obtain a region R^* with the properties described in the lemma.

It is clearly sufficient to establish the theorem for regions R which are symmetric about the lines $x = \frac{1}{2}$, $y = \frac{1}{2}$. To utilize the symmetry of R, we move the origin to the point $(\frac{1}{2}, \frac{1}{2})$. The points of Γ then become $\{(m + \frac{1}{2}, n + \frac{1}{2}) \mid m, n \in Z\}$.



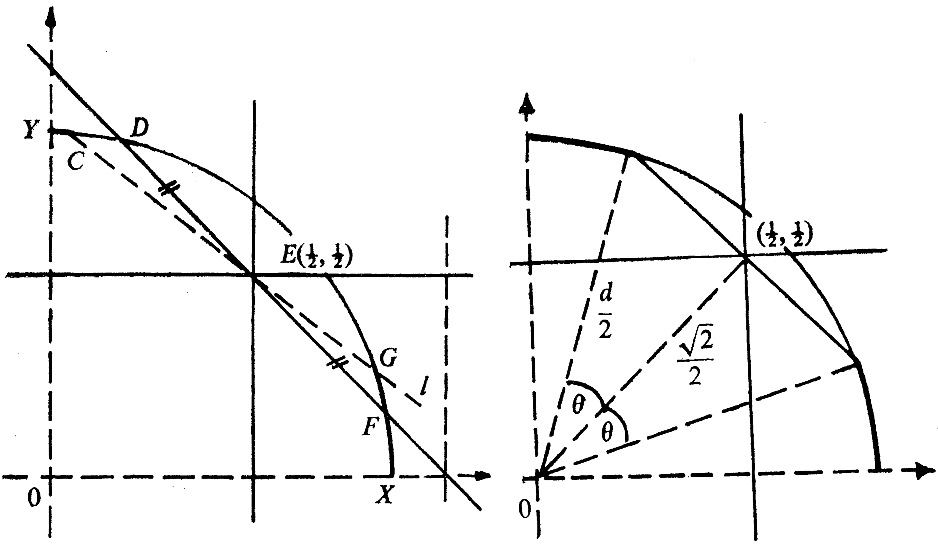
Proof of Theorem 1. Since R is centrally symmetric, it lies within the disc $x^2 + y^2 \leqq d^2/4$. If $d \leqq \sqrt{2}$, no point of Γ is interior to this disc, and a well-known property of the circle gives

$$A(R) \leqq \frac{\pi d^2}{4} \leqq \frac{\pi\sqrt{2}}{4} \cdot d < \lambda d(R) .$$

Hence we may suppose that $d > \sqrt{2}$. Consider Q, the quarter of R lying in the quadrant $x \geqq 0$, $y \geqq 0$. As Q is convex, it lies below some line l through $(\frac{1}{2}, \frac{1}{2})$ with nonpositive slope. In fact, by rotating R about O through $\pi/2$ if necessary, we may assume that the slope of l is not less than -1. Thus Q is a subset of the trapezium bounded by the axes, the line l, and the line $x = d/2$. If $d \geqq 2$, this trapezium has area at most $d/4$, and so

$$A(R) = 4A(Q) \leqq d < \lambda d(R).$$

We now assume that $\sqrt{2} \leqq d \leqq 2$. In this case, we can use a geometrical argument to show that Q attains its maximum area when the bounding line l has slope -1. For suppose l has slope greater than -1. Using the notation of the figure, $DE = EF$, and by rotating region GFE through π about E, it is clear that the area of region CDE exceeds that of region GFE. Hence the area of $OXFDY$ exceeds that of $OXGCY$. We therefore take Q to be the region bounded by the axes, the line $x + y = 1$, and two arcs of the circle $x^2 + y^2 = d^2/4$.



The line l and the circle $x^2 + y^2 = d^2/4$ determine a chord; let 2θ denote the angle subtended at the origin by this chord. Then noting that $d = \sqrt{2} \sec\theta$, we have

$$\begin{aligned}\frac{A(R)}{d(R)} &= f(\theta)\\ &= \sqrt{2}\sec\theta\left(\frac{\pi}{4} - \theta\right) + \sqrt{2}\sin\theta\\ &= \tfrac{1}{2}\sqrt{2}\sec\theta\left\{\frac{\pi}{2} - 2\theta + \sin 2\theta\right\}.\end{aligned}$$

A short calculation shows that f attains its maximum value of λ when $\sin 2\theta = \pi/2 - 2\theta$ (i.e., when θ is just under 24°). This completes the proof of the theorem.

Let r be any positive integer. In [3], Hammer shows that if R is a convex region for which $A(R) > (r/2)P(R)$, then R contains r lattice points. There is a similar result in terms of the diameter of R.

THEOREM 2. *If $A(R) > r\lambda\, d(R)$, then R contains r lattice points in its interior.*

Proof. We use induction. The result is true for $r = 1$ by Theorem 1. Let us assume that it is true for $r = k$, and let R be a region for which $A(R) > (k+1)\lambda\, d(R)$. We show that R contains $k+1$ interior lattice points.

Divide R into two convex regions R_1, R_2 such that

$$A(R_1) = \frac{k}{k+1}A(R) \text{ and } A(R_2) = \frac{1}{k+1}A(R).$$

Then

$$A(R_1) = \frac{k}{k+1}A(R) > k\lambda\, d(R) \geqq k\lambda\, d(R_1),$$

so by our assumption R_1 contains k interior lattice points.

Also,

$$A(R_2) = \frac{1}{k+1}A(R) > \lambda\, d(R) \geqq \lambda\, d(R_2),$$

so R_2 contains an interior lattice point.

Hence $R = R_1 \cup R_2$ contains $k+1$ lattice points as required.

It is interesting to compare Theorem 2 with a recent result due to Reich [**4**]: If $A(R) > r\{\frac{1}{2}P(R) + d(R)\}$, where r is any positive integer, then R contains $2r$ lattice points.

References

1. E. A. Bender, Area-perimeter relations for two-dimensional lattices, Amer. Math. Monthly, 69 (1962) 742–744.

2. H. G. Eggleston, Convexity, Cambridge University Press, New York, pp. 90–92.

3. Joseph Hammer, On a general area-perimeter relation for two-dimensional lattices, Amer. Math. Monthly, 71 (1964) 534–535.

4. Simeon Reich, Two-dimensional lattices and convex domains, this MAGAZINE, 43 (1970) 219–220.

A NON-TRUTH-FUNCTIONAL 3-VALUED LOGIC

JOHN GRANT, University of Florida

In 2-valued logic the truth-values are true (T) and false (F). For example the proposition "There exist infinitely many primes" is true, and the proposition "Every positive integer is odd" is false. But consider Goldbach's conjecture "Every even number greater than 2 is the sum of 2 primes". According to the law of the excluded middle Goldbach's conjecture is either true or false. However since we do not know which is the case, we assign this proposition a third truth-value, indeterminate (I).

In this paper we describe a non-truth-functional 3-valued logic which applies to the situation described above. We wish to contrast our system with the truth-functional 3-valued logic given in Kleene [**2**], page 334.

THEOREM 2. *If $A(R) > r\lambda\, d(R)$, then R contains r lattice points in its interior.*

Proof. We use induction. The result is true for $r = 1$ by Theorem 1. Let us assume that it is true for $r = k$, and let R be a region for which $A(R) > (k+1)\lambda\, d(R)$. We show that R contains $k+1$ interior lattice points.

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so R_2 contains an interior lattice point.

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In this paper we describe a non-truth-functional 3-valued logic which applies to the situation described above. We wish to contrast our system with the truth-functional 3-valued logic given in Kleene [2], page 334.

Kleene's system

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	I	F	I	T	I	I
T	F	F	F	T	F	F
I	T	I	I	T	T	I
I	I	I	I	I	I	I
I	F	I	F	I	I	I
F	T	T	F	T	T	F
F	I	T	F	I	T	I
F	F	T	F	F	T	T

Note that in Kleene's system in the 4 lines where p and q have the value T or F the truth-table coincides with the truth-table of 2-valued logic. However in Kleene's system there are no tautologies. For suppose that u is a propositional formula. If each propositional variable in u is given the value I, then u also gets the value I. In particular the law of the excluded middle, $p \vee \sim p$, does not always have value T. Yet we wish to say that the proposition "Either every even number greater than 2 is the sum of two primes or not every even number greater than 2 is the sum of 2 primes" is true.

Now we present our method for evaluating the truth-value of a propositional formula. Suppose that u contains the variables $p_1, \cdots, p_n$. Since this logic is 3-valued the truth-table for u has 3^n lines.

Case 1. The line in the truth-table is such that each variable takes on one of the values T or F.

Proceed as in 2-valued logic to obtain either T or F for the value of u.

Case 2. The line in the truth-table is such that some of the variables, say without loss of generality $p_1, \cdots, p_k$, take on the value I while the other variables (if any), $p_{k+1}, \cdots, p_n$, take on one of the values T or F.

Consider those 2^k lines in the truth-table where $p_1, \cdots, p_k$ take on one of the values T or F, while $p_{k+1}, \cdots, p_n$ take on their given values. Each such line is evaluated as explained under Case 1. If all of these lines have value T give u the value T; if all of them have value F give u the value F; otherwise give u the value I.

Examples

p	q	$p \vee (\sim p \rightarrow q)$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
T	I	T	T
T	F	T	T
I	T	T	T
I	I	I	T
I	F	I	T
F	T	T	T
F	I	I	T
F	F	F	T

Note that for $p \vee (\sim p \rightarrow q)$ Kleene's system yields the same result, but for $p \rightarrow (q \rightarrow p)$ Kleene's system yields a different result.

We now verify that the tautologies in our 3-valued logic are the same as the tautologies of 2-valued logic. First note that by Case 1 the truth-table for 2-valued logic is just a part of the corresponding truth-table for the 3-valued logic. It follows that if a formula is a tautology in the 3-valued logic, it is also a tautology in 2-valued logic. Conversely, if a formula u is a tautology in 2-valued logic, consider any line in the evaluation of u. If the line falls under Case 1 the value must be T. Since Case 2 depends on Case 1, if the line falls under Case 2 the value must again be T. Thus a tautology of 2-valued logic remains a tautology in the 3-valued logic.

For a logic to be truth-functional every connective must be truth-functional. For the binary connective $\wedge$ this means that given 2 formulas u and v, the value of $u \wedge v$ depends solely on the values of u and v. Now let u be the variable p, and v the variable q. If both u and v have value I, then $u \wedge v$ gets the value I. Next let u be the variable p, and v the formula $\sim p$. In this case if both u and v have value I, then $u \wedge v$ gets the value F. Thus this 3-valued logic, unlike Kleene's system, is not truth-functional.

Grant [1] developed a model theory using the 3-valued logic described in this paper instead of 2-valued logic.

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2. S. C. Kleene, Introduction to Metamathematics, Van Nostrand, Princeton, 1952.

AN IDENTITY FOR THE SUM OF DIGITS OF INTEGERS IN A GIVEN BASIS

JEAN-LOUP MAUCLAIRE, Orsay, France

Let p be an integer $\geqq 2$. We denote by $S(n)$ the sum of the digits of a nonnegative integer n in the basis p.

Let $\mathscr{N}_r(N)$ be the number of nonnegative integers $\leqq p^N - 1$ whose sum of digits is exactly equal to r, r a positive integer.

We shall prove the following result.

THEOREM: *If q is a given integer, we have for every $N \geqq 0$:*

$$N \times \left[\sum_{\nu=0}^{q-1} \mathscr{N}_\nu(n) - p \sum_{\substack{r \equiv q \bmod p \\ 0 \leqq r < q}} \mathscr{N}_r(N)\right] = q\mathscr{N}_q(N).$$

Proof: We have:

$$(*) \qquad \prod_{k=0}^{N-1} \left(\frac{1 - z^p x^{p^{k+1}}}{1 - zx^{p^k}}\right) = \sum_{n=0}^{p^N - 1} z^{S(n)} x^n = \sum_{r=0}^{N(p-1)} \mathscr{N}_{r,x}(N) z^r.$$

First, for $x = 1$:

$$(**) \qquad (1 + z + \cdots + z^{p-1})^N = \sum_{r=0}^{N(p-1)} \mathscr{N}_r(N) z^r.$$

But by derivation of (*) with respect to z:

$$\sum_{r=1}^{N(p-1)} r\mathscr{N}_{r,x}(N) z^{r-1} = \left\{\prod_{k=0}^{N-1} \left(\frac{1 - z^p x^{p^{k+1}}}{1 - zx^{p^k}}\right)\right\} \times \left(\sum_{k=0}^{N-1} \frac{x^{p^k}}{1 - zx^{p^k}} - p.\frac{z^{p-1} x^{p^{k+1}}}{1 - z^p x^{p^{k+1}}}\right).$$

Suppose $|z| < 1$, and $x = 1$; we obtain:

$$\sum_{r=1}^{N(p-1)} r\mathscr{N}_r(N) z^{r-1} = N \times \left\{\prod_{k=0}^{N-1} \left(\frac{1 - z^p}{1 - z}\right)\right\} \times \left\{\frac{1}{1 - z} - p\frac{z^{p-1}}{1 - z^p}\right\},$$

i.e., by use of (**)

$$\sum_{r=1}^{N(p-1)} r\mathscr{N}_r(N) z^{r-1}$$

$$= N \times \left(\sum_{r=0}^{N(p-1)} \mathscr{N}_r(N) z^r\right) \times (\{1 + z + z^2 + \cdots\} - pz^{p-1} \times \{1 + z^p + z^{2p} + \cdots\}).$$

Now, we remark that $\mathscr{N}_q(N)$ is equal to the coefficient of z^{q-1} in the right-hand member, and this coefficient is exactly

$$N\left(\sum_{\nu=0}^{q-1} \mathscr{N}_\nu(N) - p \times \sum_{\substack{r \equiv q \bmod p \\ 0 \leqq r < q}} \mathscr{N}_r(N)\right).$$

NOTE ON A FUNCTION SIMILAR TO $n!$

JEAN-MARIE DE KONINCK, Université Laval, Québec

Let $0^{\#} = 1$ and let $n^{\#}$ denote the least common multiple of the integers $1, 2, \cdots, n$. In a recent paper [3], D. Knutson asked if there exists a "Stirling formula" for $n^{\#}$ and furthermore what are the properties of the function $\sum_{n=0}^{\infty} x^n/n^{\#}$. The purpose of this note is to discuss these problems.

The function $n^{\#}$ is well known in number theory. In fact, if we let $\psi(n) = \sum_{p^\alpha \leqq n} \log p$, where the sums runs through all prime powers $\leqq n$, then $\psi(n) = \log n^{\#}$ [2]. Therefore an asymptotic formula for $\log n^{\#}$ can be obtained from the estimate $\psi(n) = n + o(n)$, which is equivalent to the Prime Number Theorem [1].

Concerning the series $\sum_{n=0}^{\infty} x^n/n^{\#}$, we prove that, unlike the series $\sum_{n=0}^{\infty} x^n/n!$ which converges for all real x, it has a finite radius of convergence, namely e.

Indeed, we know that $\sum_{n=0}^{\infty} a_n x^n$ has its radius of convergence equal to $1/\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$. Hence the radius of convergence R of $\sum_{n=0}^{\infty} x^n/n^{\#}$ is equal to

$$\frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\dfrac{1}{n^{\#}}}} = \frac{1}{\overline{\lim\limits_{n\to\infty}}\, e^{-(1/n)\psi(n)}}.$$

Now $\psi(n) = n + o(n)$ gives

$$\overline{\lim_{n\to\infty}}\, e^{-(1/n)\psi(n)} = \lim_{n\to\infty} e^{-(1/n)\psi(n)} = e^{-1}.$$

Therefore $R = e$.

The author wishes to thank the referee for his suggestions which simplified the presentation of this note.

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3. D. Knutson, A lemma on partitions, Amer. Math. Monthly, 79 (1972) 1111–1112.

TRIGONOMETRIC IDENTITIES

ANDY R. MAGID, The University of Oklahoma

A *trigonometric identity* is an equation between two rational functions of trigonometric functions, e.g.:

$$\frac{\tan x}{\csc x - \cot x} - \frac{\sin x}{\csc x + \cot x} = \sec x + \cos x.$$

NOTE ON A FUNCTION SIMILAR TO $n!$

JEAN-MARIE DE KONINCK, Université Laval, Québec

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The function $n^{\#}$ is well known in number theory. In fact, if we let $\psi(n) = \sum_{p^\alpha \leqq n} \log p$, where the sums runs through all prime powers $\leqq n$, then $\psi(n) = \log n^{\#}$ [2]. Therefore an asymptotic formula for $\log n^{\#}$ can be obtained from the estimate $\psi(n) = n + o(n)$, which is equivalent to the Prime Number Theorem [1].

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Indeed, we know that $\sum_{n=0}^{\infty} a_n x^n$ has its radius of convergence equal to $1/\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$. Hence the radius of convergence R of $\sum_{n=0}^{\infty} x^n/n^{\#}$ is equal to

$$\frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\dfrac{1}{n^{\#}}}} = \frac{1}{\overline{\lim\limits_{n\to\infty}}\, e^{-(1/n)\psi(n)}}.$$

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$$\frac{\tan x}{\csc x - \cot x} - \frac{\sin x}{\csc x + \cot x} = \sec x + \cos x.$$

The *ad hoc* verification of such identities is a standard exercise in elementary trigonometry (see, for example, [**1**, Chapter 8], from which the above identity comes) which most students have done, usually not in any systematic way. Thus the following theorem, which fits nicely into an undergraduate abstract algebra course, is of some interest to them:

THEOREM. *Every trigonometric identity is a consequence of* $\sin^2 x + \cos^2 x = 1$.

The proof of the theorem, which will be outlined here, uses only some elementary commutative ring theory, about at the level of [**2**, Chapter 3].

We begin the proof with some simplifications: first express all the trigonometric functions appearing in the identity in terms of $\sin x$ and $\cos x$, next clear denominators and finally subtract one side from the other. What remains is a polynomial identity of the form $f(\sin x, \cos x) = 0$. For convenience later on, assume that the polynomials have complex coefficients. Now let $\mathbb{C}$ denote the complex numbers and let A be the ring of all complex functions analytic in a neighborhood of zero. Then the theorem is equivalent to the following:

PROPOSITION. *The kernel of the ring homomorphism* $\Phi: \mathbb{C}[X, Y] \to A$ *defined by* $\Phi(X) = \cos x$ *and* $\Phi(Y) = \sin x$ *is generated by* $X^2 + Y^2 - 1$.

Now A is an integral domain (using, for example, power series expansions) and so the kernel P of Φ is a prime ideal which contains the prime ideal P_0 generated by the irreducible polynomial $X^2 + Y^2 - 1$. The proposition will be proved by showing that $P = P_0$. For experts, this is immediate: $\mathbb{C}[X, Y]$ has Krull dimension 2, so every nonzero prime properly containing P_0 is maximal. So if $P \neq P_0$, P is a maximal ideal and the image of Φ is a field. But $\Phi(X) = \sin x$ is not even a unit in A, so this is impossible. The same line of reasoning will provide an elementary proof, once the following is established:

LEMMA. *Every nonzero prime ideal of* $R = \mathbb{C}[X, Y]/P_0$ *is a maximal ideal.*

Here is a sketch of the proof: let x, y denote the images of X, Y in R. Then $x^2 + y^2 = 1$, so if $u = x + iy$, $u^{-1} = x - iy$ and $R = \mathbb{C}[u, u^{-1}]$. If I is any ideal of R, then $I = (I \cap \mathbb{C}[u])R$ by the standard argument, but since $\mathbb{C}[u]$ is Euclidean this makes $I \cap \mathbb{C}[u]$, and hence I, principal. Thus R is a principal ideal domain. The usual arguments now show that every nonzero prime of R is maximal (see, for example, [**2**, p. 109]): prime ideals are generated by prime elements and ideals generated by prime elements are maximal. This proves the lemma, hence also the proposition and the theorem.

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SETS OF COMPLEX NUMBERS

D. E. DAYKIN, Reading University, and A. WILANSKY, Lehigh University.

This expository article has benefited from several consultations with colleagues. Theorem 3 has already been published; we generalize it in Theorem 5 but feel that its elegant proof deserves repetition here.

For a finite set A of complex numbers we shall write ΣA and $\Sigma|A|$ to mean $\Sigma\{z: z \in A\}$ and $\Sigma\{|z|: z \in A\}$.

THEOREM 1. *Let A be a finite set of complex numbers. Then A has a subset B such that* $|\Sigma B| \geqq (1/6)\Sigma|A|$.

Trisect the complex plane by the negative real axis and the rays $\arg z = \pm 60°$. The three resulting subsets B_1, B_2 B_3 have the property that for some $k, \Sigma|B_k| \geqq (1/3)\Sigma|A|$. For this k, $|\Sigma B_k| \geqq \frac{1}{2}\Sigma|B_k| \geqq (1/6)\Sigma|A|$.

THEOREM 2. *The result of Theorem* 1 *is true with* 1/6 *replaced by* 1/4.

With $z = x + iy$, let $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$. Then $\Sigma|A| \leqq \Sigma|x| + \Sigma|y| = \Sigma x^+ + \Sigma x^- + \Sigma y^+ + \Sigma y^- = t_1 + t_2 + t_3 + t_4$, say, so that $t_k \geqq (\frac{1}{4})\Sigma|A|$ for some k. For example, suppose $k = 1$, the other cases being handled similarly. Let $B = \{z \in A; x > 0\}$. Then $|\Sigma B| \geqq \Sigma\{x: z \in B\} = \Sigma\{x^+: z \in B\} = \Sigma\{x^+: z \in A\} = t_1 \geqq (\frac{1}{4})\Sigma|A|$.

We shall now give a proof that $\frac{1}{4}$ can be replaced by $1/\pi$. This is the final item in our list of improvements as Theorem 4 shows that $1/\pi$ is the best possible constant. One other sharpening is still available and that is to replace the inequality by a strict one.

The following lemma and Theorem 3 are given in [1] where the proof is attributed to S. Kakutani. The results given there are about measures.

LEMMA. *Let A_1 be the set of all sums of subsets of A, and P the convex hull of A_1. Then P is a polygon of perimeter* $2\Sigma|A|$.

This result has to be interpreted generously in case A has only one (nonzero) member z. Then $A_1 = \{0, z\}$ and P is the degenerate polygon consisting of the line joining 0 to z twice. Proceeding by induction, suppose that A has $n + 1$ points. Write $A = A_n \cup \{z\}$. Then A_n yields a polygon P_n of perimeter $2\Sigma|A_n|$ by the induction hypothesis. Adding z changes P_n by translating some of its faces a distance $|z|$ thus making its perimeter $2\Sigma|A_n| + 2|z| = 2\Sigma|A|$.

THEOREM 3. *Let A be a finite set of complex numbers. Then A has a subset B such that* $|\Sigma B| > (1/\pi)\Sigma|A|$.

If this result is false the polygon P of the lemma is contained in the disc

$$D = \{z: |z| \leqq s/\pi\}$$

where $s = \Sigma|A|$. But P is a convex polygon hence its perimeter is strictly less than that of D which is $2s$, contradicting the lemma.

An interesting extension of Theorem 3 is given in [2], where $1/\pi$ is replaced by $1/3\pi$ and a restriction is placed on B.

THEOREM 4. *The result of Theorem* 3 *is false if* $1/\pi$ *is replaced by any larger number.*

Let n be a positive integer, $z = \exp(i\pi/n)$, and $A = \{1, z, z^2, \cdots z^{2n-1}\}$. It is easy to see that $B = \{1, z, z^2, \cdots, z^{n-1}\}$ is a subset of A which maximizes the left hand side of the inequality in Theorem 3. This maximum is $\left|\sum_{k=0}^{n-1} z^k\right| = \left|2/(1-z)\right| = \text{cosec}$ $(\pi/2n)$ which is asymptotically $(2n)/\pi = (1/\pi)\sum|A|$.

It is clear that B, in Theorem 3, lies in a half-plane. If we restrict B to lie in a wedge, $|\Sigma(B)|$ will of course be smaller. We show how to maximize it. Setting $\theta = \pi/2$ in Theorem 5 will yield Theorem 3. For $\theta > 0$ we set $W(\alpha, \theta) = \{z : \alpha - \theta \leqq \arg z < \alpha + \theta\}$.

THEOREM 5. *Given* $0 < \theta \leqq \pi$, *there exists* α *such that* $|\Sigma[A \cap W(\alpha, \theta)]| \geqq [(\sin\theta)/\pi]\,\Sigma|A|$. *The number* $(\sin\theta)/\pi$ *cannot be replaced by any larger one.*

We fix θ and write $W(\alpha)$ for $W(\alpha, \theta)$; s for $\Sigma|A|$. Let

$$A^* = \max\left|\{\Sigma[A \cap W(\alpha)] : 0 \leqq \alpha < 2\pi\}\right|.$$

We have to prove $A^* \geqq (s/\pi)\sin\theta$. We begin by replacing A by a set with more uniform distribution and with twice as many members. Let $A(\lambda) = (\frac{1}{2}A) \cup (\frac{1}{2}e^{i\lambda}A)$ where we count a number twice if it occurs in both sets on the right. Thus $\Sigma|A(\lambda)| = s$. We now prove that $A^* \geqq A(\lambda)^*$ for any λ. For any α, let

$$F = \Sigma[A(\lambda) \cap W(\alpha)],\ G = \Sigma[A \cap W(\alpha)],\ H = \Sigma[A \cap W(\alpha - \lambda)].$$

Then $F = \frac{1}{2}G + \frac{1}{2}e^{i\lambda}H$ and so $|F| \leqq \max(|G|, |H|)$ from which the required inequality follows. Now setting $A_0 = A$, $A_{n+1} = A_n(\pi/2^n)$ we have, by the inequality just proved, $A^* = A_0^* \geqq A_1^* \geqq A_2^* \cdots$. We compute $\lim A_n^*$ by imagining a limiting "uniform distribution," i.e , one with $|\Sigma[A \cap W(\alpha)]|$ constant. Taking $\alpha = 0$ we imagine an elementary angle dw containing one point $sdwe^{iw}/2\pi$ of A. Then $\Sigma[A \cap W(0)] = (s/2\pi)\int_{-\theta}^{\theta} e^{iw}dw = (s/\pi)\sin\theta$. Thus $\lim A_n^* = (s/\pi)\sin\theta$. A careful exposition of this proof, which would take too much space, is left to the reader.

This paper was written while the second author held a Fulbright Research Grant, on leave from Lehigh University.

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A NOTE ON ALMOST PERFECT NUMBERS

JAMES T. CROSS, The University of the South

Let $\sigma(M)$ denote the sum of the divisors of the positive integer M and let $F(M) = 2M - \sigma(M)$. M is **perfect** if $F(M) = 0$. Jerrard and Temperly [**2**] call M **perfect-minus-one** (PM1) if $F(M) = -1$ and **perfect-plus-one** (PP1) if $F(M) = 1$. M is **almost perfect** in either case. They show that if M is PM1 and 3 is not a factor of M, then M has at least seven distinct prime factors and

$$M > 1{,}382{,}511{,}906{,}801{,}025.$$

Their argument can be modified to show that this same number is a lower bound for odd PP1 numbers not divisible by 3 and that such odd PP1 numbers must have at least seven distinct prime factors.

It is easy to check that powers of 2 are PP1. We do not know whether there are other PP1 numbers. In this note we identify an infinite class of even numbers that are not PP1 and recall a theorem of Lucas that enables one to define perfect and almost perfect numbers in terms of a certain set of remainders.

PROPOSITION. *Let p denote an odd prime. If $2^{n+1} > p$, then no multiple of $2^n p$ is* PP1.

Proof. Let $L = 2^n p$. Then

$$\sigma(L) = \sigma(2^n)\sigma(p) = (2^{n+1} - 1)(p + 1) = 2^{n+1}p + 2^{n+1} - (p + 1).$$

Then $\sigma(L) = 2L + 2^{n+1} - (p + 1)$. Since $2^{n+1} > p$, $\sigma(L) \geqq 2L$ so that $F(L) \leqq 0$. If $M = KL$, M is not PP1 because $F(M) < 0$ since $F(KL) < F(L)$. (See [**2**], Proposition 1. The referee has pointed out that Proposition 1 of [**2**] follows easily from the observation that $\sigma(KL) > K\sigma(L)$ since K times any divisor of L is a divisor of KL.)

THEOREM. *For each positive integer N let $S(N)$ denote the sum of the remainders when N is divided by $1, 2, \cdots, N$. If $N > 1$, then $S(N) + \sigma(N) = S(N-1) + 2N - 1$.*

According to Dickson [**1**, p. 312], this theorem was proved by E. Lucas in 1891 in his *Théorie des nombres*, page 388. This source is not available to us. We give the following proof:

Proof. Let R_i $(i = 1, 2, \cdots,)$ denote the remainder when N is divided by i and let r_i $(i = 1, 2, \cdots,)$ denote the remainder when $N - 1$ is divided by i. If i does not divide N, then $R_i = r_i + 1$ whereas $i = r_i + 1$ when i divides N.
Then

$$S(N) + \sigma(N) = \sum_{i=1}^{N} R_i + \sum_{\substack{i=1 \\ i|N}}^{N} i = \sum_{\substack{i=1 \\ i\nmid N}}^{N-1} R_i + \sum_{\substack{i=1 \\ i|N}}^{N-1} i + N.$$

Then

$$S(N) + \sigma(N) = \sum_{i=1}^{N-1} (r_i + 1) + N = S(N-1) + 2N - 1.$$

One can redefine PM1, perfect, and PP1 numbers as those positive integers M such that $S(M) - S(M-1) = -2, -1$, and 0 respectively.

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2. R. P. Jerrard and Nicholas Temperley, Almost perfect numbers, this MAGAZINE, 46 (1973) 84.

A NOTE ON MERSENNE NUMBERS

STEVE LIGH and LARRY NEAL, University of Southwestern Louisiana

It is well known that [5, p. 17, #36] if $2^n - 1 = p$, where p is a prime, then n must be a prime. In this note we first show that if $2^n - 1 = p^m$, then $m = 1$ and n is a prime. Then we apply this result to show that if R is a near field and for each x in R, there exists a positive integer $k = p^m + 1$ such that $x^k = x$, then R is a field.

THEOREM 1. *Let m and n be positive integers and* $p \geqq 3$ *be a prime. If* $2^n - 1 = p^m$, *then* $m = 1$ *and n is a prime.*

Proof. We only need to show $m = 1$. Suppose n is even and write $n = 2k$. Then $2^n - 1 = 2^{2k} - 1 = (2^k - 1)(2^k + 1) = p^m$. Thus, for $k > 1$, p divides both $2^k + 1$ and $2^k - 1$, and hence their difference, i.e., $p = 2$, a contradiction. If $k = 1$, then $n = 2$ and clearly $m = 1$.

Now suppose n is odd and $m > 1$. Let $n = 2j + 1$. Hence $2^n - 1 = 2^{2j+1} - 1 = 2(2^{2j} - 1) + 1 = p^m$ and $2\ \sum_0^{2j-1} 2^k = p^m - 1 = (p-1)\ \sum_0^{m-1}\ p^k$. Therefore

$$(*) \qquad \sum_0^{2j-1} 2^k = \frac{p-1}{2} \sum_0^{m-1} p^k.$$

If m is even, then the right hand side of $(*)$ is even while the left hand side is odd, a contradiction.

Now suppose m is odd. We have $2^n = p^m + 1 = (p+1)\ \sum_0^{m-1}(-1)^k p^k$ and therefore

$$(**) \qquad \sum_0^{m-1}(-1)^k p^k = 2^t \quad \text{for some } t.$$

But the left hand side of $(**)$ is odd and 2^t is even, a contradiction. Thus $m = 1$ and the proof is complete.

We now apply Theorem 1 to obtain a commutativity theorem for near fields. But first a definition is needed.

DEFINITION. *An algebraic system* $(R, +, \cdot)$ *is called a near field if*

(i) $(R, +)$ *is a group,*

(ii) $(R - 0, \cdot)$ *is a group,*

One can redefine PM1, perfect, and PP1 numbers as those positive integers M such that $S(M) - S(M-1) = -2, -1$, and 0 respectively.

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$$(*) \qquad \sum_0^{2j-1} 2^k = \frac{p-1}{2} \sum_0^{m-1} p^k.$$

If m is even, then the right hand side of $(*)$ is even while the left hand side is odd, a contradiction.

Now suppose m is odd. We have $2^n = p^m + 1 = (p+1)\ \sum_0^{m-1} (-1)^k p^k$ and therefore

$$(**) \qquad \sum_0^{m-1} (-1)^k p^k = 2^t \quad \text{for some } t.$$

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DEFINITION. *An algebraic system $(R, +, \cdot)$ is called a near field if*

(i) $(R, +)$ *is a group,*

(ii) $(R - 0, \cdot)$ *is a group,*

(iii) $x(y+z)=xy+xz$ *for all* x,y,z *in* R,
(iv) $0x=0$ *for all* x *in* R.

Near fields were first studied by L. E. Dickson [**1**] in a paper in which he was concerned with proving the independence of a certain set of postulates for a finite field. That the additive group $(R,+)$ of a near field R is abelian was first shown by B. H. Neumann in [**4**]. Of course every field is an example of a near field. An example of a near field that is not a field is the system whose elements are the nine elements of the finite field $GF(3^2)$ with addition the same as $GF(3^2)$, and multiplication $*$ defined in terms of the multiplication of $GF(3^2)$ by

$$x * y = \begin{cases} xy, & \text{if } x \text{ is a square in } GF(3^2), \\ xy^3, & \text{if } x \text{ is a nonsquare in } GF(3^2). \end{cases}$$

It is well known that if a ring T has the property that for each x in T, there is an integer n such that $x^n = x$, then T is commutative [**3**, p. 73]. In particular a finite field is commutative. However the above example of near field R shows that this is not the case since $x^9 = x$ for each x in R. We now give a sufficient condition for a finite near field to be a field. But first a lemma is needed.

LEMMA. *Let R be a near field. Then*
(i) $a(-b) = -(ab) = (-a)b$ *for each* a, b *in* R,
(ii) *R is a division ring if the right distributive law holds.*

Proof. We need to show (i) only. Let a and b be arbitrary elements and 1 the identity of R. That $a\cdot 0 = 0$ and $a(-b) = -(ab)$ follow from the left distributive law. Then $a(-1) = -a$ and $(-1)(-1) = 1$. Next we show that $(-1)a = a(-1)$. Suppose that $(-1)a = a(-1) + x$, $x \neq 0$. Then

$$x = a + (-1)a = (-1)((-1)a + a) = (-1)(a + (-1)a) = (-1)x.$$

Since $x \neq 0$, it follows that $1 = -1$ and $a = a + x$. But this implies that $x = 0$, a contradiction. Using the above, it follows that $(-a)b = a(-1)b = ab(-1) = -(ab)$.

REMARK. For a discussion of finite near fields, see [**2**, p. 390].

THEOREM 2. *Let R be a finite near field and assume there is a positive integer $k = p^j + 1$, where $p \geqq 3$ is a prime and $j \geqq 1$, such that $x^k = x$ for each x in R. Then R is a field.*

Proof. Let 1 be the identity of R. Using (i) of the above lemma, we see that $(-1)(-1) = -((-1)1) = -(-1) = 1$. Since k is even, it follows that $(-1)^k = 1$. By hypothesis $(-1)^k = -1$. Hence $1 + 1 = 0$ and $x(1+1) = 0$ for each x in R. Thus $(R,+)$ is a 2-group and the order of R, $o(R)$, is 2^n for some n [**6**, p. 86].

Since R is a finite near field, $(R - 0, \cdot)$ is a finite group and if $x \neq 1$ is in R, then $x^{k-1} = 1$. Thus $x^{p^j} = 1$ implies that $(R - 0, \cdot)$ is a p-group and $o(R - 0, \cdot) = p^m$. Hence $o(R) = 2^n = p^m + 1$.

Now by Theorem 1 we see that $m = 1$. Hence $o(R - 0, \cdot) = p$ implies that $(R - 0, \cdot)$ is cyclic and thus a commutative group.

Finally we need to show that the right distributive law holds in R. But this follows easily from the fact that $(R, \cdot)$ is commutative. Hence R is a field.

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ON THE EXISTENCE OF ABSOLUTE PRIMES

T. N. BHARGAVA and P. H. DOYLE, Kent State University

An absolute prime base ten is a prime number such that every permutation of its digits is also prime. Our method of establishing the existence of nontrivial examples consists of utilizing a computer program search among the first million positive integers. We present a list of those found in this range. We believe that the list may not be exhaustive even for primes that involve only two distinct digits in their presentation. Indeed the problem of existence of absolute primes that employ just two distinct digits could be, to the best of our knowledge, as difficult as the enumeration of the Mersenne primes.

An absolute prime with more than one digit can employ only $1, 3, 7$, or 9 as digits. We exhibit the list of absolute primes we found, and we prove that no absolute prime may utilize all four of the numbers $1, 3, 7, 9$ as digits. We wish to thank the referee for a thorough study of our first draft.

We found the following absolute primes. They are: $2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 199, 311, 337, 373, 733, 919, 991$. Note that twelve on this list have no repeated digits.

THEOREM. *There exists no absolute prime utilizing all four digits* $1, 3, 7,$ *and* 9.

Proof. Upon division by 7 the numbers 1379, 1793, 3719, 7913, 7193, 3197, and 7139 have remainders $0, 1, 2, 3, 4, 5$, and 6, respectively. Hence, an integer N having all these digits in it may be permuted into $N_1 = K + 1379$. If one divides K by 7, let $K = 7q + r$ $(0 \leqq r < 7)$. Then 1379 may be permuted so that its remainder is $7 - r$ upon division by 7 and so N may be permuted to the form

$$N_2 = 7q + r + 7r + (7 - r)$$

and so 7 divides N_2.

Now by Theorem 1 we see that $m = 1$. Hence $o(R - 0, \cdot) = p$ implies that $(R - 0, \cdot)$ is cyclic and thus a commutative group.

Finally we need to show that the right distributive law holds in R. But this follows easily from the fact that $(R, \cdot)$ is commutative. Hence R is a field.

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1. L. E. Dickson, On finite algebras, Nachr. Akad. Wiss. Gottingen, (1905) 358–393.
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ON THE EXISTENCE OF ABSOLUTE PRIMES

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MATHEMATICAL SOCIAL SCIENCE — AN EARLY EXAMPLE

"In this day of moral uncertainty ... there may be some of my readers who would be grateful for a definite, quantitative basis for the calculation of morals. For their peace of mind and in the hope that they will find it practically useful, I present one of Hutcheson's* formulas:

$$M = (B + S_e)A$$

$$u = (H + S_u)A,$$

therefore

$$M - u = (BA + S_eA) - (HA + S_uA)$$

where

M = moment of good

u = moment of evil

$M - u$ = balance of good

B = benevolence

S_e = enlightened self-love

H = hatred or social malevolence

S_u = unenlightened self-love

A = agent's ability."

Edward W. Hall in *Modern Science and Human Values*, D. Van Nostrand Co. Inc. (1956), p. 387.

* Francis Hutcheson, professor of moral philosophy at the University of Glasgow in the first half of the eighteenth century.

NOTES AND COMMENTS

Arthur Marshall points out that the conjecture of Erdös and Strauss that $4/n$ can be written as the sum of the reciprocals of three or fewer integers, which S. W. Golomb mentions in his paper *On representing an integer as the harmonic mean of integers* in the November 1973 issue, is not only true for n even as Golomb notes but also for $n \equiv 3 \pmod 4$ since

$$\frac{4}{n} = \frac{4}{n+1} + \frac{4}{n(n+1)}, \text{ for } n \equiv 9 \pmod{12} \text{ since } \frac{4}{n} = \frac{1}{n} + \frac{3}{n}$$

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and for

$$n \equiv 5 \pmod{12} \text{ since } \frac{4}{n} = \frac{1}{n} + \frac{3}{n+1} + \frac{3}{n(n+1)}.$$

Hence only the case $n \equiv 1 \pmod{12}$ remains open.

In connection with the article *A 'good' generalization of the Euler-Fermat theorem* by Roger Osborn in the January 1974 issue, David Singmaster has called attention to his paper *A maximal generalization of Fermat's theorem* in the March 1966 issue. As the title implies, that paper presents a better generalization than the "good" one given by Osborn. It also includes applications and a summary of the history of the problem. Singmaster's paper was unaccountably overlooked both by Professor Osborn and by the editors.

John P. Hoyt writes regarding the article *On the Steiner-Lehmus theorem* by Mordechai Lewin in the March 1974 issue, "School Science and Mathematics for June 1939 contained an article 'The Lehmus-Steiner Theorem' by David L. MacKay in which the author gave the history and over a dozen proofs including the one by F. G. Hesse (1874) which I consider direct involving only first degree line relationships and which I think should serve as a starting point for anyone interested in proofs of this theorem."

Albert Wilansky comments as follows on *Contagious properties* by Chew in the March 1974 issue: A T_B space (called KC space in [**1**], [**2**], [**3**]) is one in which compact sets are closed. Clearly compactness is contagious (P is called *contagious* if whenever S has P, so does $\bar{S}$) in a KC space. The question is asked (p. 85) whether compactness is contagious in a T_1 space. *The answer is no*; see [**1**], p. 334, table 3, column 2, line 10; p. 87 #114.

The question is asked, p. 85, whether countable compactness is contagious. *The answer is no* even in a (completely regular) topological group; see [**1**], p. 339, column 2, lines 4, 5; p. 367, note 34.

The question is asked, p. 86, whether completeness is contagious in metric spaces. We interpret this as follows: a topological space is called complete if it is homeomorphic to a complete metric space. Then *the answer is no*. Let X be the union of the open unit disc D in the complex plane and $\{e^{i\theta}: \theta \text{ rational}\}$. Then D is complete and dense but X is not complete. See [**1**], p. 173 #206, p. 181 #11.

The question is asked whether all contagious properties are continuous invariants. Here is an example. Let P be the property: "X is homeomorphic with $[0, 1]$." Then P is contagious (in T_2 spaces) but is not a continuous invariant.

Theorem 6 (Pseudocompactness is a continuous invariant) may be found in [**1**], p. 352, column 1, line 3; p. 59 #104.

1. A. Wilansky, Topology for analysis, Xerox College Publishing Company, 1970.
2. ——— Between T_1 and T_2, Amer. Math. Monthly, 74(1967) 261–266.
3. ——— Life without T_2, Amer. Math. Monthly, 77(1970) 157–161.

Additional comments on *Contagious properties*:

(a) From Kwang Chul Ha and Lawrence E. Spence: The property of being a Baire space is contagious but is not a continuous invariant.

(b) From James Chew: In the proof of Theorem 4 the open intervals $(-N, N)$ should be closed intervals $[-N, N]$, the function g used in the proof of Theorem 6 needs to be continuous, and Theorem 7 should read "In the class of T_1 spaces, T_i ($i = 1\frac{2}{3}, 2, 3, 4$) is not contagious."

P. M. Gibson writes regarding the paper *Norm preserving operators in decomposable tensors* by Richard Bronson in the March 1973 issue. In a paper *Unitary and orthogonal transformations on matrices* which will appear in the November 1974 issue of the Siam Journal of Applied Mathematics, Gibson proves the proposition for which Bronson presented a counterexample. Bronson's counterexample is not valid. He applied the norm for a real space to a complex space.

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use.*

What is Mathematical Logic? By J. N. Crossley, J. C. Stillwell, C. J. Ash, N. H. Williams, and C. Brickhill. Oxford University, New York, 1972. 77 pp., $1.95 paper, $4.50 cloth.

This book was adapted from a series of lectures given by the authors. It discusses, without proof, a rather broad collection of topics in first order logic. The first chapter consists of a historical survey, some classical results with sketches of proofs and remarks on the material to be covered later. The second chapter describes the predicate calculus and proceeds through a rather thorough sketch of Henkin's proof of the Completeness Theorem. The third chapter gives an introduction to model theory including the Compactness and Löwenheim-Skolem Theorems. In the fourth chapter, Turing machines are defined and the idea of a partial recursive function is developed. The fifth chapter sketches the proof of the Incompleteness Theorem. The last chapter treats set theory proceeding from an intuitive justification of the axioms to sketches of Godel's proof of the consistency of the continuum hypothesis and Cohen's proof of the independence of the generalized continuum hypothesis. The book is 77 pages long.

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On the other hand, there are a number of flaws. There is a bit of a tendency to oversimplify some details in the treatment. For example, logic seems to be tacitly identified with first order logic and the statement "the Löwenheim-Skolem Theorem says that every theory has countable models" is made without qualification in chapter 1. It seems to be implied on page 34 that a Turing machine requires extra symbols for markers and hence needs more than one nonblank symbol in its alphabet. The remarks about completeness and incompleteness in chapter 1 could easily puzzle a serious student. The pictorial "cone" description of models of set theory on page 74 is treated too briefly to make sense to the reader who has not thought about the material before. Of course, as in most books, there are also several minor misprints.

Overall, *What is Mathematical Logic?* can be recommended to several types of readers. For the student who has completed a course in logic it can be suggested to gain a broader perspective on the material he has studied and to gain a better view about how its parts relate to one another. For the mathematician or the student with some mathematical sophistication, the book provides a brief synopsis of the field providing hints for future reading and a chance to become familiar with the basic ideas. For the more casual reader who is not particularly interested in understanding all the details and can keep up with the pace, the book provides an honest, reasonably careful description of what logic is all about. For the teacher a number of excellent pedagogical ideas are displayed. With the caveats above it would seem a worthwhile addition to the libraries of most of those seriously interested in mathematics.

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Mathematics, Its Spirit and Evolution. By John R. Durbin. Allyn and Bacon, Boston, Massachusetts, 1973. xii + 321 pages. $10.50.

As stated in the preface, this text is the author's answer to the question of "how the spirit of mathematics can best be conveyed to those with a limited background in the subject." Towards this end, chapters on non-Euclidean geometry, probability, cardinal numbers, groups, and analysis are presented. The discussion of these topics, together with a reasonable amount of the history of mathematics as "leavening," provides much in the way of fascinating reading. The writing is lucid, figures are legible and helpful, and answers to selected problems are included.

Returning to the preface, it is also stated that "The reader is assumed to have knowledge of minimal high school mathematics (a course in each of geometry and algebra)." I have serious doubts that such minimal prerequisites can assure the level of mathematical maturity that seems to be needed by students to successfully study from this text. The exposition and development appear to me to assume a degree of mathematical sophistication that one might hope to find in students with at least three years of high school mathematics. Even if each topic to be covered is many difficult ideas and the authors convey a feeling of real concern that their readers follow the reasoning. Basically the material is presented in a technically correct manner while exploiting first rate pedagogical methods to put across some of the harder ideas.

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H. R. HYATT, Los Angeles Pierce College

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

Editor's Note: In order to provide for more time for solvers to prepare solutions and thereby reduce the number of late solvers, especially for persons outside the continental United States, the time between the proposal and the deadline for solutions is being delayed one issue. Consequently the solutions to Problems 880 through 886 will be postponed until the November 1974 issue. Hereafter the deadline dates will be as follows: January-February issue, August 1; March-April issue, October 1; May-June issue, December 1; September-ber-October issue, April 1; November-December issue, June, 1.

To be considered for publication, solutions should be mailed before April 1, 1975.

PROPOSALS

908. *Proposed by J. A. H. Hunter, Toronto, Canada.*

We define N as an integer with $(2n + 1)$ digits, the first digit not a zero. Then say N is represented as $(A)(B)$, (A) having n digits, (B) having $(n + 1)$ digits.

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When (A) and (B) are interchanged the result is equal to $8N$. What is the smallest value of N that meets this requirement?

909. *Proposed by Dennis R. Lichtman and James L. Murphy, California State College, San Bernardino.*

Let ϕ denote Euler's phi-function. Find the positive integers n such that n and $\phi(n)$ are relative primes.

910. *Proposed by L. Carlitz, Duke University.*

Let P be a point in the interior of the triangle ABC and let r_1, r_2, r_3 denote the distances from P to the sides of ABC. Let a, b, c denote the sides and r the radius of the incircle of ABC. Show that

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} \geqq \frac{2s}{r}, \tag{1}$$

$$ar_1^2 + br_2^2 + cr_3^2 \geqq 2r^2 s, \tag{2}$$

$$(s-a)r_2r_3 + (s-b)r_3r_1 + (s-c)r_1r_2 \leqq r^2 s, \tag{3}$$

$$ar_1^2 + br_2^2 + cr_3^2 + (s-a)r_2r_3 + (s-b)r_3r_1 + (s-c)r_1r_2 \geqq 3r^2 s, \tag{4}$$

where $2s = a + b + c$. In each case there is equality if and only if P is the incenter of ABC.

911. *Proposed by Charles W. Trigg, San Diego, California.*

On a standard cubical die, the three faces around one vertex are numbered 1, 2, 3 in order counterclockwise. The digits 4, 5, 6 are distributed on the other three faces so that the sum of the digits on each pair of opposite faces is 7.

When the numbers on the three faces around each vertex are added, the minimum sum is 6 and the maximum sum is 15. This is a span of ten integers, but there are only eight vertices. Without actually adding up the digits around the other six vertices, determine what the two numbers missing among the sums are.

912. *Proposed by Michael O'Rourke, University of Wisconsin — Parkside.*

Defining three functions of the natural numbers

$$f_1(n) = \sum_{i=0}^{[n/2]} (-1)^i 2^{n-2i} \binom{n-i}{i},$$

$$f_2(n) = \sum_{i=0}^{(n-1)/2} (-1)^i 2^{n-(2i+1)} \binom{n-i}{i}\binom{n-2i}{1},$$

$$f_3(n) = \sum_{i=0}^{(n-2)/2} (-1)^i 2^{n-(2i+2)} \binom{n-i}{i}\binom{n-2i}{2},$$

show that

$$f_1(n) = n + 1,$$

$$f_2(n) = \binom{n+2}{3},$$

$$f_3(n) = \binom{n+3}{5},$$

913. *Proposed by J. Garfunkel, Forest Hills High School, Flushing, New York.*

Triangle $A_1B_1C_1$ is inscribed in a circle. The medians are drawn and extended to the circle meeting the circle at points $A_2B_2C_2$. The medians of triangle $A_2B_2C_2$ are likewise drawn and extended to the circle to points $A_3B_3C_3$ and so on. Prove that triangle $A_nB_nC_n$ becomes equilateral as $n \to \infty$ (and very rapidly).

914. *Proposed by Murray S. Klamkin, Ford Motor Company.*

If for any n of a given $n + 1$ integral weights, there exists a balance of them on a two pan balance where a fixed number of weights are placed on one pan and the remainder on the other pan, prove that the weights are all equal.

Late Solvers

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QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q598. If A flips n fair coins and B flips $n+1$ fair coins, prove that the probability that B flips more heads than A is 1/2.

[*Submitted by Steven R. Conrad*]

Q599. If $a(n)$ denotes the exponent of the prime p in the prime factorization of n, determine the sum

$$S(m) = a(1) + a(2) + \cdots + a(p^m).$$

[*Submitted by Murray S. Klamkin*]

Q600. A box contains a shiny pennies and b dull ones. A penny is drawn at random without replacement until the kth shiny penny is drawn. Let X be the number of the draw on which the kth shiny penny is drawn. It is easy to show that when $k = a$,

$$E(X) = a(a+b+1)/(a+1).$$

Use this result to find $E(X)$ for $k = 1$.

[*Submitted by John P. Hoyt*]

Q601. If $\{z_i\}$ and $\{w_i\}$, $(w_1 \neq z_1 \neq z_2)$ denote complex numbers such that

$$\begin{vmatrix} z_1 & z_4 & 1 \\ z_2 & z_3 & 1 \\ w_1 & w_2 & 1 \end{vmatrix} = \begin{vmatrix} z_1 & z_1 & 1 \\ z_2 & z_3 & 1 \\ w_1 & w_3 & 1 \end{vmatrix} = \begin{vmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ w_1 & w_3 & 1 \end{vmatrix} = 0,$$

prove that $z_2 - z_1 = z_3 - z_4$.

[*Submitted by Murray S. Klamkin*]

Q602. If $0 < A_i < \pi$ and $\sum_{i=1}^n A_i = 2\pi$ show that $\sum_{i=1}^n \sin A_i \leqq n \sin(2\pi/n)$ and $\sum_{i=1}^n \tan(A_i/2) \geqq n \tan(\pi/n)$.

[*Submitted by Allan Wayne*]

(Answers on page 224.)

ANSWERS

A598. Clearly, B either flips more heads than A or more tails than A but cannot do both. Since one case is indistinguishable from the other, the probability is 1/2. It is interesting to note that if B had $n+2$ coins, then the probability is a function of n, but that when B has $n+1$ coins the probability is independent of n.

A599. $S(m+1) - S(m) = a(p^m + 1) + a(p^m + 2) + \cdots + a(p^{m+1})$. Since, $a(pq) = 1 + a(q)$ and $a(r) = 0$ if $p \nmid r$, $S(m+1) - S(m) = (p-1)p^{m-1} + a(p^{m-1}+1) + a(p^{m-1}+2) + \cdots + a(p^{m-1} + (p-1)p^{m-1})$

or

$$S(m+1) - 2S(m) + S(m-1) = (p-1)p^{m-1}.$$

It now follows easily that

$$S(m) = \frac{p^m - 1}{p - 1}.$$

Remark: The special case $p = 2$ was given as a problem on a recent Dutch mathematical Olympiad.

A600. Think of the pennies stacked in a pile before drawing (such as a deck of cards). Then the ath shiny penny measured from one end of the stack is the first shiny penny measured from the other end. Hence $E(X) = (a+b+1) - a(a+b+1)/(a+1) = (a+b+1)/(a+1)$. An obvious extension of this process enables one to find $E(X)$ for any k.

A601. Solving for w_2 and w_3 from the first two determinants being zero and substituting into the third one, we obtain

$$(z_2 + z_4 - z_1 - z_3)(w_1 - z_1)(w_1 - z_2) = 0.$$

Thus, it is necessary and sufficient that

$$z_2 - z_1 = z_3 - z_4.$$

The sufficiency condition is equivalent to the following geometric theorem: *If ABCD is a parallelogram and ABX, DCY and ACZ are directly similar triangles, then also XYZ* $\sim$ *ABX* (this is given as an exercise in T. M. MacRobert, *Functions of a Complex Variable*, Macmillan, London, 1950, p. 277).

A602. For a given circle, the set of inscribed (circumscribed) polygons, those with maximum (minimum) area, are the regular polygons.

(Quickies on page 241.)

THE GREATER METROPOLITAN NEW YORK MATH FAIR

The Greater Metropolitan New York Math Fair will be held on April 13, 1975, at Pace University in Manhattan. It is a mathematical competition for students who have completed, or are currently taking, mathematics at the eleventh year level or higher in the public, private and parochial high schools in the New York City, Westchester, Putnam, Dutchess, or Rockland Counties. Exceptional students who do not meet these requirements but wish to submit papers for consideration by the Fair Committee are welcome to do so.

Further details and application forms may be obtained from: Dr. Theresa J. Barz, Secretary, Math Fair Committee, Department of Mathematics and Computer Science, St. John's University, Jamaica, NY 11439.
